CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS : A REVIEW

DISSERTATION

SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE AWARD OF THE DEGREE OF

Master of Philosophy
in
Statistics

By

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ALIGARH (INDIA)

2011

DEDICATED

TO

MY

PARENTS

AND

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ACKNOWLEDGEMENT

All praises due to the Almighty Allah, the most Beneficent, the most Merciful, who bestowed upon me the courage, patience and strength to embark upon this work and carry it to its completion and admires to His last and great Prophet, (PBUH), whose moments of life provided me with the strength to run fast on the path of success.

It is good fortune and a matter of pride and privilege for me to have the esteemed supervision of ***Dr. Haseeb Athar,*** *who has inculcated in me the interest and inspiration to undertake research in the field of order statistics. It is only his personal influence, expert guidance and boundless support that enabled me to complete the work in the present form.*

I place on record my special sense of gratitude to ***Prof. A. H. Khan*** *former Dean, Faculty of Science and Ex-Chairman, Department of Statistics & O.R., for his suggestion, moral support and wholehearted cooperation.*

I am also grateful to ***Prof. M.M. Khalid,*** *Chairman, Department of Statistics & O.R. ,* ***Prof. M.Z. Khan, Prof. M. Yaqub, Prof. Q. M. Ali, Prof. A. A. Khan, Dr. R. U. Khan, Dr. Mohd. Faizan*** *and other teachers of the Dept. for their encouragement and co-operation during this work.*

It is my pleasure to express my deep sense of appreciation to all my colleagues and friends.

*My special thanks are due to **Mr. Ziaul Haque** , **Mr. Mohd. Izhar Khan, Mr. Imtiyaz Ahmad Shah** and **Mr. Nayabuddin** for their valuable guidance, cooperation and sympathetic behavior at all stages of this work.*

Thanks are also due to all non-teaching staff members of the Department for their help and cooperation.

I would like to express my deep sense of indebtedness to my father, who has been dreaming for quite a long time for my success. He has done everything possible to see me at this place. He has always helped me in good and bad times alike to keep me focused towards my goal. The completion of this dissertation would have been rather impossible without his wholehearted support, encouragement and sacrifices during the tough and easier phases of this project. I feel shortage of words to pay thanks to him. I would also like to express my special thanks to my brothers and sisters for their best wishes.

I would like to take this opportunity to express my gracious concern towards all those people who have helped in various ways throughout the development stages of this dissertation.

Zuber Akhter

Date: 5.08.2011 *(Zuber Akhter)*

PREFACE

Order random variates like order statistics (os), record value and generalized order statistics (gos) have significant role in characterizing the probability distributions. Characterization results are located on the borderline between probability theory and mathematical statistics, and utilize numerous classical tools of mathematical analysis. The useful characterizations results are those which shed light on modeling consequences of certain distributional assumptions and those which have potential for development of hypothesis tests for model assumptions.

Characterization theorems are the only methods, which allow us to avoid the subjective choice of F and lead to the accurate $F(x)$ through simple properties.

The present dissertation entitled **"Characterization of Probability Distributions: A review"** is a brief collection of the work done so far on the topic. I have tried my best to include sufficient, up to date and relevant materials in the systematic way, which are contained in five chapters.

Chapters I is introductory in nature, deals with the concept of order statistics, record values, generalized order statistics and dual generalized order statistics and some useful results based on these ordered random scheme which are needed in subsequent chapters.

Chapter II deals with characterization of probability distributions through moments, conditional expectation and distributional properties of order statistics for some specific distributions as well as for general class of distributions.

Chapter III is based on characterization of probability distributions through record values, where moments, conditional expectation and distributional properties of record statistics have been considered and characterization of some specific distributions as well as for general class of distributions is shown.

Chapter IV embodies results on characterization of distributions through moments and distributional property of generalized order statistics.

Chapter V, the characterization results are shown through moments and distributional property of dual (lower) generalized order statistics.

In the end, a comprehensive list of references referred into this dissertation is given.

PRELIMINARIES AND BASIC CONCEPTS

1. INTRODUCTION

In this chapter a brief review of the concepts and results used in subsequent chapters have been presented. Section 2 deals with basic definition and distribution theory of order statistics. In this section concept of conditional and truncated distributions and some important results based on order statistics are also given. In Section 3, concept of record values and record times is given whereas Section 4 and 5 deal with generalized order statistics and lower (dual) generalized order statistics respectively. Some useful results used in subsequent chapters based on *gos* and *dgos* are also presented in these sections. In section 1.7, some basic continuous distributions are discussed.

2. ORDER STATISTICS

2.1 Definition

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a continuous population having probability density function (*pdf*) $f(x)$ and distribution function (*df*) $F(x)$. Let they be arranged in ascending order of magnitude as

$$X_{1:n} \le X_{2:n} \le \ldots\ldots \le X_{r:n} \le \ldots\ldots \le X_{n:n}$$

then $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ are collectively called the order statistics of the sample and $X_{r:n}\ (r = 1, 2, \ldots, n)$ is called the $r-th$ order statistic of the sample. $X_{1:n} = \min(X_1, X_2, \ldots, X_n)$ and $X_{n:n} = \max(X_1, X_2, \ldots, X_n)$ are called extreme order statistics or the smallest and the largest order statistics.

David and Nagaraja (2003) is the basic book on order statistics dealing in detail with its different aspects. Asymptotic theory of extremes and related developments of order statistics are well described in an applausive work of Galambos (1987). Also, references may be made to Sarhan and Greenberg

(1962), Balakrishnan and Cohen (1991), Arnold *et al.* (1992) and the references therein.

2.2 Distribution of order statistics

Here in this section we will discuss the basic distribution theory of order statistics by assuming that population is absolutely continuous.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a continuous population having probability density function (*pdf*) $f(x)$ and distribution function (*df*) $F(x)$. Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the corresponding order statistics, then the *pdf* of $X_{r:n}$, the $r-th$ order statistic is given by (David and Nagaraja, 2003)

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x) \; , \; -\infty < x < \infty \qquad (2.1)$$

The $pdf's$ of smallest and largest order statistics are,

$$f_{1:n}(x) = n[1-F(x)]^{n-1} f(x) \; ; \; -\infty < x < \infty \qquad (2.2)$$

$$f_{n:n}(x) = n[F(x)]^{n-1} f(x) \; ; \; -\infty < x < \infty \qquad (2.3)$$

The df of $X_{r:n}$ is given by

$$F_{r:n}(x) = P(X_{r:n} \le x)$$

$$= P \text{ (at least } r \text{ of } X_1, X_2, \ldots, X_n \text{ are less than or equal to } x)$$

$$= \sum_{i=r}^{n} P(\text{exactly i of } X_1, X_2, \ldots, X_n \text{ are less than or equal to } x)$$

$$= \sum_{i=r}^{n} \binom{n}{i} [F(x)]^i [1-F(x)]^{n-i} \; ; \; -\infty < x < \infty \qquad (2.4)$$

$$= \frac{n!}{(r-1)!(n-r)!} \int_0^{F(x)} u^{r-1}(1-u)^{n-r} du \qquad (2.5)$$

$$= \mathrm{I}_{F(x)}(r, n-r+1) \qquad (2.6)$$

RHS is obtained by the relationship between binomial sums and incomplete beta function. It may be expressed in negative binomial sums as (Khan, 1991)

$$F_{r:n}(x) = \sum_{i=0}^{n-r} \binom{n-1-i}{r-1} [F(x)]^r [1-F(x)]^{n-r-i}; \quad -\infty < x < \infty \tag{2.7}$$

For continuous case the *pdf* of $X_{r:n}$ may also be obtained by differentiating (2.5) *w.r.t.* x.

From the density function given in (2.1), we may obtain the $k-th$ moment of $X_{r:n}$ as below:

$$\mu_{r:n}^{(k)} = E[X_{r:n}^k] = \int_{-\infty}^{\infty} x^k f_{r:n}(x)dx \tag{2.8}$$

Further, if $E[\xi(X_{r:n})], 1 \le r \le n$ denote the expectation of function of $r-th$ order statistic and define the inverse function of F by

$F^{-1}(t) = \inf\{x : F(x) \ge t\};\ t \in (0,1)$, then

$$E[\xi(X_{r:n})] = C_{r:n} \int_0^1 \xi(F^{-1}(u)) u^{r-1} (1-u)^{n-r} du \tag{2.9}$$

where , $C_{r:n} = \dfrac{n!}{(r-1)!(n-r)!}$

The joint *pdf* of $X_{r:n}$, $X_{s:n}$, $1 \le r < s \le n$ is given by

$$f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} [F(x)]^{r-1}$$

$$\times [F(y)-F(x)]^{s-r-1} [1-F(y)]^{n-s} f(x) f(y)\,; -\infty < x < y < \infty \tag{2.10}$$

The joint *df* of $X_{r:n}$ and $X_{s:n}$, $(1 \le r < s \le n)$ can be obtained as follows:

$$\begin{aligned} F_{r,s:n}(x,y) &= P(X_{r:n} \le x, X_{s:n} \le y) \\ &= P(\text{at least } r \text{ of } X_1, X_2, \dots, X_n \text{ are at most } x \\ &\qquad \text{and at least } s \text{ of } X_1, X_2, \dots, X_n \text{ are at most } y) \\ &= \sum_{j=s}^{n} \sum_{i=r}^{j} P(\text{exactly } i \text{ of } X_1, X_2, \dots, X_n \text{ are at most } x \\ &\qquad \text{and exactly } j \text{ of } X_1, X_2, \dots, X_n \text{ are at most y}) \end{aligned}$$

$$=\sum_{j=s}^{n}\sum_{i=r}^{j}\frac{n!}{i!(j-i)!(n-j)!}[F(x)]^{i}[F(y)-F(x)]^{j-i}[1-F(y)]^{n-j} \tag{2.11}$$

We can write the joint df of $X_{r:n}$ and $X_{s:n}$ in (2.11) equivalently as:

$$F_{r,s:n}(x,y)=\frac{n!}{(r-1)!(s-r-1)!(n-s)!}\int_{0}^{F(x)}\int_{0}^{F(y)}u^{r-1}(v-u)^{s-r-1}\times(1-v)^{n-s}\,dudv$$

$$=I_{F(x),F(y)}(r,s-r,n-s+1)\,;\,-\infty<x<y<\infty \tag{2.12}$$

which is incomplete bivariate beta function.

It may be noted that for $x\geq y$

$$F_{r,s:n}(x,y)=F_{s:n}(y) \tag{2.13}$$

The product moments of the $j-th$ and $k-th$ order of $X_{r:n}$ and $X_{s:n}$ respectively, $(1\leq r<s\leq n)$ is given by:

$$\mu_{r,s:n}^{(j,k)}=E[X_{r:n}^{j}X_{s:n}^{k}]=\iint\limits_{-\infty<x<y<\infty}x^{j}\,y^{k}f_{r,s:n}(x,y)dxdy \tag{2.14}$$

Let $E[\xi(X_{r:n},X_{s:n})], 1\leq r<s\leq n$ denotes the expectation of function of two order statistics, then we have

$$E[\xi(X_{r:n},X_{s:n})]=C_{r,s:n}\iint\limits_{0\leq u<v\leq 1}\xi(F^{-1}(u),F^{-1}(v))u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s}\,du\,dv \tag{2.15}$$

where, $C_{r,s:n}=\dfrac{n!}{(r-1)!(s-r-1)!(n-s)!}$

In general, the joint pdf of $X_{i_1:n},X_{i_2:n},\ldots,X_{i_k:n}$ for $1\leq i_1<i_2<\ldots<i_k\leq n$ is given by

$$f_{i_1,i_2,\ldots,i_k:n}(x_{i_1:n},x_{i_2:n},\ldots,x_{i_k:n})$$

$$=n!\left\{\prod_{j=1}^{k}f(x_{i_j})\right\}\prod_{j=0}^{k}\left\{\frac{[F(x_{i_{j+1}})-F(x_{i_j})]^{i_{j+1}-i_j-1}}{(i_{j+1}-i_j-1)!}\right\}$$

$$-\infty<x_{i_1}<x_{i_2}<\ldots<x_{i_k}<\infty \tag{2.16}$$

where $x_0=-\infty, x_{k+1}=+\infty, i_0=0, i_{k+1}=n+1$

Remarks:

1. The ranking of random variables $X_1, X_2, ..., X_n$ is preserved under any monotonic increasing transformation of the random variables.
2. Regarding the probability integral transformation, if $X_{r:n}, 1 \le r \le n$, are the order statistics from a continuous distribution $F(x)$, then the transformation $U_{r:n} = F(X_{r:n})$ produces a random variable which is the $r-th$ order statistic from a uniform distribution on $U(0,1)$.
3. Even if $X_1, X_2, ..., X_n$ are independent random variables, order statistics are not independent random variables.
4. Let $X_1, X_2, ..., X_n$ be *iid* random variables from a continuous distribution, then the set of order statistics $\{X_{1:n}, X_{2:n}, ..., X_{n:n}\}$ is both sufficient and complete (Lehmann, 1986).
5. Let X be a continuous random variable with $E[X_{r:n}] = \alpha_{r:n}$,
   a) If $\alpha = E(X)$ exits then $\alpha_{r:n}$ exists, but converse is not necessarily true. That is, $\alpha_{r:n}$ may exist for certain (but not all) values of r, even though α does not exist.
   b) $\alpha_{r:n}$ for all n determine the distribution completely.

2.3 Truncated and conditional distribution of order statistics

Let X be a continuous random variable having *pdf* $f(x)$ and *df* $F(x)$ in the interval $[-\infty, \infty]$.

Let $\int_{-\infty}^{Q_1} f(x)dx = Q$ and $\int_{-\infty}^{P_1} f(x)dx = P$ (2.17)

where Q_1 and P_1 are known constants. Then doubly truncated *pdf* of X is given by:

$$\frac{f(x)}{P-Q}; \; x \in (Q_1, P_1) \tag{2.18}$$

and the corresponding *cdf* is given by

$$\frac{F(x)-Q}{P-Q}\ ;\ x \in (Q_1, P_1) \tag{2.19}$$

The lower and upper truncation points are Q_1, P_1 respectively; the degrees of truncation are Q (from below) and $1-P$ (from above). If we put $Q=0$, the distribution will be truncated to the right. Similarly, for $P=1$, the distribution will be truncated to the left. Whereas for $Q=0, P=1$, we get the non truncated distribution. Truncated distributions are useful in finding the conditional distributions of order statistics.

2.4 Some important results

Result 1 (David and Nagaraja, 2003): Let $X_1, X_2, \ldots, X_n$ be a random sample from an absolutely continuous population with the *df* $F(x)$ and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{r:n}$, given that $X_{s:n} = y$ for $s > r$, is the same as the distribution of the $r-th$ order statistic obtained from a sample of size $(s-1)$ from a population whose distribution is truncated on the right at y.

Result 2 (David and Nagaraja, 2003): Let $X_1, X_2, \ldots, X_n$ be a random sample from an absolutely continuous population with the *df* $F(x)$ and *pdf* $f(x)$, and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{s:n}$, given that $X_{r:n} = x$ for $r < s$, is the same as the distribution of the $(s-r)-th$ order statistic obtained from a sample of size $(n-r)$ from a population whose distribution is truncated on the left at x.

Result 3 (David and Nagaraja, 2003): Let $X_1, X_2, \ldots, X_n$ be a random sample from an absolutely continuous population with *df* $F(x)$ and *pdf* $f(x)$, and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{s:n}$ given that $X_{r:n} = x$ and

$X_{k:n} = z$ for $1 \le r < s < k \le n$, is the same as the distribution of the $(s-r)-th$ order statistic obtained from a sample of size $(k-r-1)$ from a population whose distribution is truncated on the left at x and on the right at z.

Result 4: Order statistics in a sample from a continuous distribution form a Markov chain, that is

$$f(X_{k:n} \mid X_{1:n} = x_1, \ldots, X_{r:n} = x_r, \ldots X_{s:n} = x_s, \ldots X_{n:n} = x_n)$$
$$= f(X_{k:n} \mid X_{r:n} = x_r,\, X_{s:n} = x_s)$$

So, because of the Markovian properties of order statistics, it is of no use to condition it on more than two order statistics.

Result 5 (Ali and Khan, 1997): Let $g(x)$ be a Borel measurable function of x in the interval $[\alpha, \beta]$ then, for $1 \le r \le n$, $n = 1, 2, \ldots$

(i) $E[g(X_{r:n})] - E[g(X_{r-1:n-1})]$

$$= \binom{n-1}{r-1} \int_{Q_1}^{P_1} g'(x)[F(x)]^{r-1}[1 - F(x)]^{n-r+1}\, dx. \tag{2.20}$$

(ii) $E[g(X_{r:n})] - E[g(X_{r-1:n})]$

$$= \binom{n}{r-1} \int_{Q_1}^{P_1} g'(x)[F(x)]^{r-1}[1 - F(x)]^{n-r+1}\, dx. \tag{2.21}$$

(iii) $E[g(X_{r-1:n-1})] - E[g(X_{r-1:n})]$

$$= \binom{n-1}{r-2} \int_{Q_1}^{P_1} g'(x)[F(x)]^{r-1}[1 - F(x)]^{n-r+1}\, dx. \tag{2.22}$$

In view of (2.20), (2.21) and (2.22), we have

$$(n-r+1)E[g(X_{r-1:n})] + (r-1)E[g(X_{r:n})] = nE[g(X_{r-1:n-1})]. \tag{2.23}$$

At $g(x) = x$ in (2.23), we get the well known relation established by (David and Nagaraja, 2003).

Result 6 (Ali and Khan, 1998): If $g(\ )$ is a Borel measurable function from $\Re^2$ to $\Re$, then for $1 \le r < s \le n$, $n = 1, 2, \cdots$

$$E[g(X_{r:n}, X_{s:n})] - E[g(X_{r:n}, X_{s-1:n})]$$

$$= \frac{C_{r,s:n}}{(n-s+1)} \iint_{Q_1 \le x < y \le P_1} \frac{\partial}{\partial y} g(x,y)[F(x)]^{r-1}$$

$$\times [F(y) - F(x)]^{s-r-1}[1-F(y)]^{n-s+1} f(x) dx\, dy. \tag{2.24}$$

Result 7 (Khan *et al.*, 2001): If $g(\ )$ is a Borel measurable function from $\Re^2$ to $\Re$, then for $1 \le r < s \le n$, $n = 1, 2, \cdots$

$$E[g(X_{r:n}, X_{s:n})] - E[g(X_{r-1:n}, X_{s:n})]$$

$$= \frac{C_{r,s:n}}{(s-r)} \int_{Q_1}^{P_1} \int_{x}^{P_1} \frac{\partial}{\partial x} g(x,y)[F(x)]^{r-1}$$

$$\times [F(y) - F(x)]^{s-r}[1-F(y)]^{n-s} f(y) dy\, dx. \tag{2.25}$$

Result 8 (Rao and Shanbhag, 1994)):

We have Rao and Shanbhag (1994) result on *Choquet-Deny* type functional equation as below:

Let

$$\int_{R_+} G(u+v)\mu(du) = G(v) + c^* \quad \text{a.e. [L] for } u \in R_+ = [0, \infty) \tag{2.26}$$

where $G : R_+ \to R = (-\infty, \infty)$ is locally integrable Borel measurable function and μ is a σ – finite measure on R_+ with $\mu(\{0\}) < 1$ then

$$G(x) = \begin{cases} \gamma + \alpha'[1 - \exp(\eta x)] & \text{a.e.[L] if } \eta \ne 0 \\ \gamma + \beta' x & \text{a.e.[L] if } \eta = 0 \end{cases} \tag{2.27}$$

where α', β', γ are constants and η is such that

$$\int_{R_+} e^{\eta x} \mu(dx) = 1 \tag{2.28}$$

3. RECORD VALUES AND RECORD TIMES

3.1 Definition

Suppose that $X_1, X_2, \ldots, X_n$ is a sequence of independent and identically distributed random variables with *df* $F(x)$. Let $Y_n = \max(\min)\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. We say X_j is an upper (lower) record values of $\{X_n, n \geq 1\}$, if $Y_j > (<) Y_{j-1}, j > 1$. By definition X_1 is an upper as well as lower record values. One can transform the upper record by replacing the original sequence of $\{X_j\}$ by $\{-X_{j,} j \geq 1\}$ or if $P(X_i > 0) = 1$ for all i by $\left\{\frac{1}{X_i}, i \geq 1\right\}$, the lower record value of this sequence will correspond to the upper record values of the original sequence (Ahsanullah, 1995)

The indices at which upper record values occur are given by the record times $\{U_{(n)}\}, n > 0$. That is $X_{U(n)}$ is the $n-th$ upper record, where $U_{(n)} = \min\{j \mid j > U_{(n-1)}, X_j > X_{U(n-1)}, n > 1\}$ and $U_{(n)} = 1$. The distribution of $U_{(n)}, n \geq 1$ does not depend on F. Further, we will denote $L_{(n)}$ as the indices where the lower record values occur. By assumption $U_{(1)} = L_{(1)} = 1$. The distribution of $L_{(n)}$ also does not depend on F.

Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recoding them: e.g. Olympic records or world records in sports.

Record values are defined by Chandler (1952) as a model of successive extremes in a sequence of identically and independent random variables. It may also be helpful as a model for successively largest insurance claims in non-life insurance, for highest water-levels or highest temperatures. Record values are also useful in reliability theory.

To be precise, record values are defined by means of record times. That is, those times have to be described at which successively largest values appear.

Chandler (1952) shows several properties of record values and notes their Markovian structure Two recent books on records by Ahsanullah (1995) and Arnold *et al.* (1998) are worth mentioning.

3.2 Distribution of record values

Let $R(x)$ be a continuous function of x with $R(x) = -\ln \overline{F}(x)$ and $0 < \overline{F}(x) = 1 - F(x)$, where 'ln' is the natural logarithm.

If we define $F_n(x)$ as the *df* of $X_{U(n)}$ for $n \geq 1$, then we have (Ahsanullah, 1995)

$$F_n(x) = P(X_{U(n)} \leq x)$$

$$= \int_{-\infty}^{x} \frac{R^{n-1}(u)}{(n-1)!} dF(u), \quad -\infty < x < \infty \tag{3.1}$$

and the *pdf* $f_n(x)$ of $X_{U(n)}$ is

$$f_n(x) = \frac{R^{n-1}(x)}{(n-1)!} f(x), \qquad -\infty < x < \infty \tag{3.2}$$

The joint *pdf* of $X_{U(i)}$ and $X_{U(j)}$ is

$$f_{i,j}(x_i, x_j) = \frac{(R(x_i))^{i-1}}{(i-1)!} r(x_i) \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} f(x_j)$$
$$-\infty < x_i < x_j < \infty \tag{3.3}$$

The joint *pdf* of the n record values $X_{U(1)}, X_{U(2)}, \cdots, X_{U(n)}$ is given by

$$f_{1,2,\ldots,n}(x_1, x_2, \ldots, x_n) = r(x_1) r(x_2) \ldots r(x_{n-1}) f(x_n),$$
$$-\infty < x_1 < x_2 < \ldots < x_{n-1} < x_n < \infty \tag{3.4}$$

where $\quad r(x) = \dfrac{dR(x)}{dx} \; = \; \dfrac{f(x)}{1-F(x)}, \quad 0 < F(x) < 1$

is known as hazard rate .

In particular at $i=1,\ \ j=n.$ we have

$$f_{1,n}(x_{1,}x_n)=\ r(x_1)\frac{(R(x_n)-R(x_1))^{n-2}}{(n-2)!}f(x_n),\ \ -\infty<x_1<x_2<\infty.$$

The conditional distribution of $X_{U(j)}\mid X_{U(i)}=x_i$ is

$$f(X_{U(j)}\mid X_{U(i)}=x_i)=\frac{f_{ij}(x_i,x_j)}{f_i(x_i)}$$

$$=\frac{(R(x_j)-R(x_i))^{j-i-1}}{(j-i-1)!}\frac{f(x_j)}{1-F(x_i)},\ \ -\infty<x_i<x_j<\infty \qquad (3.5)$$

and for $X_{U(i)}\mid X_{U(j)}=x_j$ is

$$f(X_{U(i)}\mid X_{U(j)}=x_j)$$

$$=\frac{(j-1)!}{(i-1)!(j-i-1)!}\left[\frac{R(x_i)}{R(x_j)}\right]^{i-1}\left[1-\frac{R(x_i)}{R(x_j)}\right]^{j-i-1}\frac{r(x_i)}{R(x_j)}$$

$$-\infty<x_i<x_{i+1}<\infty \qquad (3.6)$$

3.3 k-Records

In some situations record values themselves are viewed as 'outlier' and hence second or third largest values are of special interest. Insurance claims in some non-life insurance can be used as an example.

Let $X_1,X_2,\cdots,X_n$ be an identically and independent sequence of random variables with a continuous distribution function $F(x)$ and let k be a positive integer.

Then the random variables $L^{(k)}(n)$ is given by (Kamps, 1995)

$$L^{(k)}(n)=1$$

$$L^{(k)}(n+1)=\min\{j\in N;X_{j,\,j+k-1}>X_{L^{(k)}(n),\,L^{(k)}(n)+k-1}\},n\in N,$$

are called $k-th$ record times and the quantities $X_{L^{(k)}(n)},n\in N$ are called $k-th$ record values or k - records.

We can obtain ordinary record values at $k = 1$.

The joint density of the k − records $X_{L^{(k)}(1)}, \cdots, X_{L^{(k)}(r)}$ is given as

$$f_{X_{L^{(k)}(1)}, \cdots X_{L^{(k)}(r)}}(x_1, \ldots, x_r)$$

$$= k^r \left(\prod_{i=1}^{r-1} \frac{f(x_i)}{1 - F(x_i)} \right) [1 - F(x_r)]^{k-1} f(x_r) \tag{3.7}$$

and the marginal densities and marginal distribution functions are given by:

$$f_{X_{L^{(k)}(r)}}(x) = \frac{k^r}{(r-1)!} [R(x)]^{r-1} [1 - F(x)]^{k-1} f(x) \tag{3.8}$$

and

$$F_{X_{L^{(k)}(r)}}(x) = 1 - [1 - F(x)]^k \sum_{j=0}^{r-1} \frac{1}{j!} [k\, R(x)]^j \tag{3.9}$$

4. GENERALIZED ORDER STATISTICS

The concept of generalized order statistics (*gos*) have been introduced and extensively studied by Kamps (1995). A variety of ordered models of random variables is contained in this concept.

4.1 Definition

Let $n \geq 2$ be a given integer and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$, $k \geq 1$ be the parameters such that

$$\gamma_i = k + n - i + \sum_{j=i}^{n-i} m_j > 0 \quad \text{for} \quad 1 \leq i \leq n-1.$$

Then $X(1, n, \tilde{m}, k), X(2, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ are called generalized order statistics if their joint probability density function (*pdf*) has the form

$$k \left(\prod_{j=1}^{n-1} \gamma_j \right) \prod_{i=1}^{n-1} [1 - F(x_i)]^{m_i} f(x_i) [1 - F(x_n)]^{k-1} f(x_n) \tag{4.1}$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1)$ of $\Re^n$

with absolutely continuous distribution function (*df*) $F()$ with probability density function (*pdf*) $f()$. The model of generalized order statistics contains as special cases such as ordinary order statistics $(\gamma_i = n-i+1; i=1,2,\dots,n$ i.e. $m_1 = m_2 = \dots = m_{n-1} = 0, k=1)$, $k-th$ record values $(\gamma_i = k$ i.e. $m_1 = m_2 = \dots = m_{n-1} = -1,\ k \in N)$, sequential order statistics $(\gamma_i = (n-i+1)\alpha_i; \alpha_1, \alpha_2, \dots, \alpha_n > 0)$, order statistics with non-integral sample size $(\gamma_i = \alpha - i + 1; \alpha > 0)$, Pfeifer's record values $(\gamma_i = \beta_i;\ \beta_1, \beta_2, \dots, \beta_n > 0)$ and progressive type II censored order statistics $(m_i \in N_0, k \in N)$ are obtained [Kamps (1995), Kamps and Cramer (2001)].

4.2 Distribution of generalized order statistics

Case I: $m_1 = m_2 = \dots = m_{n-1} = m$

The marginal density of the $r-th$ generalized order statistic (*gos*) is given by [Kamps, 1995]

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[1-F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) \tag{4.2}$$

and the joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$ is

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y)$$

$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!}[1-F(x)]^m \ g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[1-F(y)]^{\gamma_s - 1} f(x) f(y) \tag{4.3}$$

where $C_{r-1} = \prod_{i=1}^{r} \gamma_i$, $\gamma_i = k + (n-i)(m+1)$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \ne -1 \\ -\log(1-x), & m = -1 \end{cases}$$

$$g_m(x) = \int_0^x (1-t)^m dt = h_m(x) - h_m(0),\ x \in [0,1)$$

The conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k)=x$, $1 \le r < s \le n$ is given by

$$f_{X(s,n,m,k)|X(r,n,m,k)}(y \mid x)$$
$$= \frac{C_{s-1}}{(s-r-1)!C_{r-1}} \frac{[h_m(F(y))-h_m(F(x))]^{s-r-1}[1-F(y)]^{\gamma_s -1} f(y)}{[1-F(x)]^{\gamma_r -1}}, \quad x<y \tag{4.4}$$

and the conditional *pdf* of $X(r,n,m,k)$ given $X(s,n,m,k)=y$, $1 \le r < s \le n$ is

$$f_{X(r,n,m,k)|X(s,n,m,k)}(x \mid y) = \frac{(s-1)!(m+1)}{(r-1)!\,(s-r-1)!}$$
$$\times \frac{[\bar{F}(x)]^m [1-(\bar{F}(x))^{m+1}]^{r-1} [(\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1}]^{s-r-1}}{[1-(\bar{F}(y))^{m+1}]^{s-1}} f(x), \quad x<y \tag{4.5}$$

Case II: $\gamma_i \ne \gamma_j$; $i,j = 1,2,\dots,n-1$

The *pdf* of $X(r,n,\tilde{m},k)$ is [Kamps and Cramer, 2001]

$$f_{X(r,n,\tilde{m},k)}(x) = C_{r-1}\, f(x) \sum_{i=1}^{r} a_i(r)[1-F(x)]^{\gamma_i -1} \tag{4.6}$$

and the joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \le r < s \le n$ is

$$f_{X(r,n,\tilde{m},k)\,X(s,n,\tilde{m},k)}(x,y) = C_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}$$
$$\times \left[\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\right] \frac{f(x)}{(1-F(x))} \frac{f(y)}{(1-F(y))} \tag{4.7}$$

where $\quad C_{r-1} = \prod_{i=1}^{r} \gamma_i, \ \gamma_i = k+n-i+M_i$

$$a_i(r) = \prod_{\substack{j=1 \\ j \ne i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \ 1 \le i \le r \le n$$

and $\quad a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \ne i}}^{s} \frac{1}{(\gamma_j - \gamma_i)}, \ r+1 \le i \le s \le n$

Thus, the conditional *pdf* of $X(s,n,\tilde{m},k)$ given $X(r,n,\tilde{m},k)=x$, $1 \le r < s \le n$ is given by

$$f_{X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k)}(y \mid x)$$

$$= \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[\frac{1-F(y)}{1-F(x)}\right]^{\gamma_i} \frac{f(y)}{[1-F(y)]}, \; x \le y \tag{4.8}$$

and the conditional *pdf* of $X(r,n,\tilde{m},k)$ given $X(s,n,\tilde{m},k)=y$, $1 \le r < s \le n$ is given by

$$f_{X(r,n,\tilde{m},k)|X(s,n,\tilde{m},k)}(x \mid y)$$

$$= \frac{\sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i} \left\{\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}\right\} \frac{f(x)}{\bar{F}(x)}}{\left\{\sum_{i=1}^{s} a_i(s)[\bar{F}(y)]^{\gamma_i}\right\}} \tag{4.9}$$

4.3 Some important results

Result 1: (Athar and Islam, 2004)

Let $\xi(x)$ is a measurable function of x which is differentiable, then for any arbitrary distribution function F and $2 \le r \le n,\ n \ge 2$ and $k = 1, 2, \ldots\ldots$, following relations hold:

Case I: $m_1 = m_2 = \ldots = m_{n-1} = m$

$$E[\xi\{X(r,n,m,k)\}] - E[\xi\{X(r-1,n,m,k)\}]$$

$$= \frac{C_{r-2}}{(r-1)!} \int_{\alpha}^{\beta} \xi'(x)[1-F(x)]^{\gamma_r} g_m^{r-1}(F(x))\,dx \tag{4.10}$$

$$E[\xi\{X(r-1,n,m,k)\}] - E[\xi\{X(r-1,n-1,m,k)\}]$$

$$= -\frac{(m+1)}{\gamma_1} \frac{C_{r-2}^{(n)}}{(r-2)!} \int_{\alpha}^{\beta} \xi'(x)[1-F(x)]^{\gamma_r} g_m^{r-1}(F(x))dx \tag{4.11}$$

$$E[\xi\{X(r,n,m,k)\}]-E[\xi\{X(r-1,n-1,m,k)\}]$$

$$=\frac{C_{r-2}^{(n-1)}}{(r-1)!}\int_{\alpha}^{\beta}\xi'(x)[1-F(x)]^{\gamma_r}\,g_m^{r-1}(F(x))dx \tag{4.12}$$

Case II: $m_i \neq m_j(\gamma_i \neq \gamma_j);\ i \neq j = 1,2,\ldots n-1$

$$E[\xi\{X(r,n,\tilde{m},k)\}]-E[\xi\{X(r-1,n,\tilde{m},k)\}]$$

$$=C_{r-2}\int_{\alpha}^{\beta}\xi'(x)\sum_{i=1}^{r}a_i(r)[1-F(x)]^{\gamma_i}\,dx \tag{4.13}$$

$$E[\xi\{X(r-1,n,\tilde{m},k)\}]-E[\xi\{X(r-1,n-1,\tilde{m}^*,k)\}]$$

$$=-\frac{\{(r-1)+\sum_{j=1}^{r-1}m_j\}}{\gamma_1}C_{r-2}\int_{\alpha}^{\beta}\xi'(x)\sum_{i=1}^{r}a_i(r)[1-F(x)]^{\gamma_i}\,dx \tag{4.14}$$

$$E[\xi\{X(r,n,\tilde{m},k)\}]-E[\xi\{X(r-1,n-1,\tilde{m}^*,k)\}]$$

$$=\frac{\gamma_r}{\gamma_1}C_{r-2}\int_{\alpha}^{\beta}\xi'(x)\sum_{i=1}^{r}a_i(r)[1-F(x)]^{\gamma_i}\,dx \tag{4.15}$$

where $\tilde{m}^* = (m_2, m_3, \ldots, m_{n-1}) \in \Re^{n-1}$.

Result 2: (Athar and Islam, 2004)

For $1 \le r < s \le n-1$, $n \ge 2$ and $k = 1,2,\ldots..$

Case I: $m_1 = m_2 = \ldots = m_{n-1} = m$

$$E[\xi\{X(r,n,m,k),X(s,n,m,k)\}]-E[\xi\{X(r,n,m,k),X(s-1,n,m,k)\}]$$

$$=\frac{C_{s-2}}{(r-1)!(s-r-1)!}\int_{\alpha}^{\beta}\int_{x}^{\beta}\frac{\partial}{\partial y}\xi(x,y)[1-F(x)]^m f(x)g_m^{r-1}(F(x))$$

$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[1-F(y)]^{\gamma_s}\,dydx \tag{4.16}$$

Case II: $m_i \neq m_j (\gamma_i \neq \gamma_j); i \neq j = 1,2,\ldots n-1$

$$E[\xi\{X(r,n,\tilde{m},k), X(s,n,\tilde{m},k)\}] - E[\xi\{X(r,n,\tilde{m},k), X(s-1,n,\tilde{m},k)\}]$$

$$= C_{s-2} \iint_{\alpha \le x < y \le \beta} \frac{\partial}{\partial y}\xi(x,y)\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right]$$
$$\times \left[\sum_{i=1}^{r} a_i(r)\left(1-F(x)\right)^{\gamma_i}\right]\frac{f(x)}{1-F(x)}dy\,dx \qquad (4.17)$$

where, $\xi(x,y) = \xi_1(x).\xi_2(y)$

5. Lower (Dual) Generalized Order Statistics

Generalized order statistics can be easily applicable in practice problems except that when $F()$ is so called inverse distribution function. So the concept of lower generalized order statistics is needed. Pawlas and Szynal (2001) introduced the concept of lower generalized order statistics (*lgos*) to enable a common approach to descending ordered *rv*'s like reversed order statistics and lower record values. The work of Burkschat *et al.* (2003) may also be seen for dual (lower) generalized order statistics.

5.1 Definition

Let $n \ge 2$ be a given integer and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, $k \ge 1$ be the parameters such that

$$\gamma_i = k + n - i + \sum_{j=i}^{n-i} m_j > 0 \quad \text{for} \quad 1 \le i \le n-1.$$

By the lower generalized order statistics from an absolutely continuous distribution function $F()$ with the density function $f()$ we mean random variables $X'(1,n,\tilde{m},k),\ldots,X'(n,n,\tilde{m},k)$ having joint density function of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^{m_i} f(x_i)\right)[F(x_n)]^{k-1} f(x_n) \qquad (5.1)$$

for $F^{-1}(1) > x_1 \ge x_2 \ge \ldots \ge x_n > F^{-1}(0)$.

Here it may be noted that the joint density (4.1) is obtained by replacing $1-F(x)$ with $F(x)$ in (3.1).

5.2 Distribution of lower (dual) generalized order statistics

Case I: $m_1 = m_2 = \ldots = m_{n-1} = m$.

The density function of $r-th$ lower generalized order statistic is given by

$$f_{X'(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) \tag{5.2}$$

The joint density function of $r-th$ and $s-th$ lower generalized order statistics is

$$f_{X'(r,n,m,k),X'(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x) g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s - 1} f(y), \ \alpha \le y < x \le \beta, \tag{5.3}$$

where,

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1} x^{m+1}, & m \neq -1 \\ -\log x, & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \quad x \in [0,1).$$

Case II: $\gamma_i \neq \gamma_j, \ i,j = 1,2,\cdots,n-1$.

The pdf of $r-th$ lower generalized order statistic is given by

$$f_{X'(r,n,\tilde{m},k)}(x) = C_{r-1} \ f(x) \sum_{i=1}^{r} a_i(r) [F(x)]^{\gamma_i - 1} \tag{5.4}$$

and the joint pdf of $r-th$ and $s-th$ lower generalized order statistics is

$$f_{X'(r,n,\tilde{m},k),X'(s,n,\tilde{m},k)}(x,y) = C_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[\frac{F(y)}{F(x)}\right]^{\gamma_i}$$

$$\times \sum_{i=1}^{r} a_i(r) [F(x)]^{\gamma_i} \frac{f(x)}{F(x)} \frac{f(y)}{F(y)}, \quad \alpha \le y < x \le \beta, \tag{5.5}$$

where,

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \; \gamma_i = k + n - i + M_i$$

$$a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \quad 1 \le i \le r \le n$$

and $$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \neq i}}^{s} \frac{1}{(\gamma_j - \gamma_i)}, \; r + 1 \le i \le s \le n.$$

Further, the conditional *pdf* of $X'(j,n,\tilde{m},k)$ given $X'(r,n,\tilde{m},k) = x$ and $X'(s,n,\tilde{m},k) = y$, $1 \le r < j < s \le n$, is given by

$$f_{X'(j,n,\tilde{m},k)|X'(r,n,\tilde{m},k),X'(s,n,\tilde{m},k)}(t \mid x, y) =$$

$$\frac{\left[\sum_{i=r+1}^{j} a_i^{(r)}(j) \left\{\frac{F(t)}{F(x)}\right\}^{\gamma_i}\right]\left[\sum_{i=j+1}^{s} a_i^{(j)}(s) \left\{\frac{F(y)}{F(t)}\right\}^{\gamma_i}\right] \frac{f(t)}{F(t)}}{\left[\sum_{i=r+1}^{s} a_i^{(r)}(s) \left\{\frac{F(y)}{F(x)}\right\}^{\gamma_i}\right]} \tag{5.6}$$

6. Some Continuous Distributions

6.1 Pareto distribution

A random variable X is said to have the Pareto distribution if its probability density function $(pdf)\, f(x)$ and distribution function $(df)\, F(x)$ are of the form given below:

$$f(x) = p\lambda^p x^{-(p+1)}; \quad \lambda \le x < \infty; \quad \lambda, p > 0$$

$$F(x) = 1 - \lambda^p x^{-p}; \quad \lambda \le x < \infty; \quad \lambda, p > 0$$

Many socio-economic and naturally occurring quantities are distributed according to Pareto law. For example, distribution of city population sizes, personal income etc.

6.2 Power function distribution

A random variable X is said to have a power function distribution if its *pdf* and *df* are of the form given below:

$$f(x) = p\lambda^{-p}x^{p-1}; \quad 0 \le x < \lambda; \quad \lambda, p > 0$$

$$F(x) = \lambda^{-p}x^{p}; \quad 0 \le x < \lambda; \quad \lambda, p > 0$$

The power function distribution is used to approximate representation of the lower tail of the distribution of random variable having fixed lower bound. It may be noted that if X has a power function distribution, then $Y = \frac{1}{X}$ has a Pareto distribution.

6.3 Beta distribution

i) Beta distribution of first kind

A random variable X is said to have the beta distribution of first kind if its *pdf* is of the form

$$f(x) = \frac{1}{B(p,q)} x^{p-1}(1-x)^{q-1}; \quad 0 \le x \le 1, \quad p, q > 0$$

Beta distribution arises as the distribution of an ordered variable from a rectangular distribution. Suppose $X_{r:n}$ is an ordered sample from $U(0,1)$, then $X_{r:n}$ is distributed as $B(r, n-r+1)$. The standard rectangular distribution $R(0,1)$ is the special case of beta distribution of first kind obtained by putting the exponents p and q equal to 1. If $q = 1$, the distribution reduces to power function distribution.

ii) Beta distribution of second kind

The continuous random variable X which is distributed according to probability law:

$$f(x)=\frac{1}{B(p,q)}\frac{x^{p-1}}{(1+x)^{p+q}}\quad (p,q)>0,\ 0\le x<\infty$$

is known as a beta variate of the second kind with parameters p and q.

Remark 6.3.1: Beta distribution of second kind reduces to beta distribution of first kind if we replace $1+x$ by $\frac{1}{y}$.

Usage: The Beta distribution is one of the most frequently employed distributions to fit theoretical distributions. Beta distribution may be applied directly to the analysis of Markov processes with "uncertain" transition probabilities.

6.4 Weibull distribution

A random variable X is said to have a Weibull distribution if its *pdf* is given by:

$$f(x)=\theta p x^{p-1}e^{-\theta x^{p}};\qquad 0\le x<\infty;\ \theta>0,\ p>0$$

and the *df* is given by

$$F(x)=1-e^{-\theta x^{p}};\qquad 0\le x<\infty;\ \theta>0,\ p>0$$

Remark 6.4.1: If we put $p=1$ in Weibull distribution, we get the *pdf* of exponential distribution.

Remark 6.4.2: If we put $p=2$, it gives *pdf* of Rayleigh distribution.

Remark 6.4.3: If X has a Weibull distribution, then the *pdf* of

$Y=-p\log\left(\frac{X}{\alpha}\right)$ is

$$f(y)=e^{-y}e^{-e^{-y}}$$

which is a form of an Extreme Value distribution.

Remark 6.4.4: The *pdf* and the *cdf* of **inverse Weibull distribution** is given by

$$f(x) = \theta\, p\, x^{-(p+1)}\, e^{-\theta x^{-p}}, \quad 0 \le x < \infty;\ \theta > 0,\ p > 0$$

$$F(x) = e^{-\theta\, x^{-p}}, \ 0 \le x < \infty;\ \theta > 0,\ p > 0$$

Usage: Weibull distribution is widely used in reliability and quality control. The distribution is also useful in cases where the conditions of strict randomness of exponential distribution are not satisfied. It is sometimes used as a tolerance distribution in the analysis of quantal response data.

6.5 Exponential distribution

A random variable X is said to have an exponential distribution if its *pdf* is given by

$$f(x) = \theta e^{-\theta x}; \quad 0 \le x < \infty;\ \theta > 0$$

and the *df* is given by

$$F(x) = 1 - e^{-\theta x};\ 0 \le x < \infty;\ \theta > 0$$

Usage: The exponential distribution plays an important role in describing a large class of phenomena particularly in the area of reliability theory. The exponential distribution has many other applications. In fact, whenever a continuous random variable X assuming non-negative values satisfies the assumption,

$$P(X > s + t \mid X > s) = P(X > t) \text{ for all } s \text{ and } t,$$

then X will have an exponential distribution. This is particularly a very appropriate failure law when present does not depend on the past, for example, in studying the life of a bulb etc.

6.6 Rectangular distribution

A random variable X is said to have a rectangular distribution if its *pdf* is given by

$$f(x) = \frac{1}{\lambda - \beta}; \ \beta \le x \le \lambda$$

and the *df* is given by

$$F(x) = \frac{x - \beta}{\lambda - \beta}; \ \beta \le x \le \lambda .$$

The standard rectangular distribution $R(0,1)$ is obtained by putting $\beta = 0$ and $\lambda = 1$. It is noted that every distribution function $F(x)$ follows rectangular distribution $R(0,1)$. This distribution is used in "rounding off" errors, probability integral transformation, random number generation, traffic flow, generation of normal, exponential distribution etc.

6.7 Burr distribution

Let X be a continuous random variable, then different forms of cumulative distribution function of X are listed below (Johnson and Kotz, 1970):

i) $F(x) = x, \quad 0 < x < 1$

ii) $F(x) = (1 + e^{-x})^{-k}, \ -\infty < x < \infty$

iii) $F(x) = (1 + x^{-c})^{-k}, \ 0 \le x < \infty$

iv) $F(x) = \left[1 + \left(\frac{c - x}{x}\right)^{1/c}\right]^{-k}, \ 0 \le k \le c$

v) $F(x) = [1 + ce^{-\tan x}]^{-k}, \ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

vi) $F(x) = [1 + ce^{-k \sinh x}]^{-k}, \quad -\infty < x < \infty$

vii) $F(x) = 2^{-k}(1 + \tanh x)^{k}, \qquad -\infty < x < \infty$

viii) $F(x)=\left(\frac{2}{\pi}\tan^{-1}e^{x}\right), \qquad -\infty<x<\infty$

ix) $F(x)=1-\frac{2}{c[(1+e^{x})^{k}-1]+2}, \quad -\infty<x<\infty$

x) $F(x)=(1+e^{-x^{2}})^{k}, \quad 0\le x<\infty$

xi) $F(x)=\left(x-\frac{1}{2\pi}\sin 2\pi x\right)^{k}, \quad 0\le x\le 1$

xii) $F(x)=1-(1+x^{c})^{-k}, \quad 0\le x<\infty$

where k and c are positive parameters.

Special attention is given to type XII, whose *pdf* is given as:

$$f(x)=kcx^{c-1}(1+x^{c})^{-(k+1)};\ 0\le x<\infty;\ k,c>0$$

This distribution is frequently used for the purpose of graduation and in reliability theory. At $c=1$, it is called **Lomax distribution** whereas at $k=1$, it is known as **Log-logistic distribution**.

6.8 Cauchy distribution

The special form of the Pearson type VII distribution, with *pdf*

$$f(x)=\frac{1}{\pi\lambda}\frac{1}{[1+\{(x-\theta)/\lambda\}^{2}]} \qquad -\infty<x<\infty;\ \lambda>0;\ -\infty<\theta<\infty$$

is called the Cauchy distribution.

The *cdf* is given by

$$F(x)=\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{x-\theta}{\lambda}\right) -\infty<x<\infty;\ \lambda>0;\ -\infty<\theta<\infty$$

The distribution is symmetrical about $x=\theta$. The distribution does not possess finite moments of order greater than or equal to 1, and so does not possess a finite expected value or standard deviation. However, θ and λ are location and scale parameters, respectively, and may be regarded as being analogous to mean and standard deviation.

There is no standard form of the Cauchy distribution, as it is not possible to standardize without using (finite) values of mean and standard deviation, which does not exist in this case. However, a standard form is obtained by putting $\theta = 0, \lambda = 1$. The standard probability density function is given by

$$f(x) = \frac{1}{\pi} \frac{1}{(1+x^2)} \qquad -\infty < x < \infty$$

and the standard cumulative distribution function is

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} x \qquad -\infty < x < \infty .$$

CHAPTER II

CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS THROUGH ORDER STATISTICS

1. INTRODUCTION

Various developments in characterization theory dealing with order statistics and related topics have been reviewed by a number of authors including Ahsanullah (1978), Balakrishnan and Basu (1995), David (1995), Galambos and Kotz (1978), Nagaraja (1988) and others.

Moments of order statistics are extensively used in characterization of specific distributions. Govindrajulu (1975), Lin (1987) characterized the exponential, uniform, logistic and pareto distributions, using the relationships between two moments of order statistics. Lin (1988) and Kamps (1991) used recurrence relations of moments of single order statistics to characterize some specific distributions.

Grudzien and Szynal (1995) characterized the uniform distribution in terms of moments of order statistics when the sample size is random. Grudzien and szynal (1999) characterized the power distributions via moments of order statistics.

Ali and Khan (1998b) characterized general class of distributions by an application of Müntz–Szász theorem.

Khan and Abu-Salih (1989) characterized a general class of distribution through conditional expectations of function order statistics by means of the relations

$$E[h(X_{r+1:n}) \mid X_{r:n} = x] = a^* h(x) + b^*$$

and $$E[h(X_{r:n}) \mid X_{r+1:n} = x] = a_1^* h(x) + b_1^* .$$

Wesolowski and Ahsanullah (1997) characterized distributions by the regression of non-adjacent order statistics through the relation

$$E[X_{r+2:n} \mid X_{r:n} = x] = a\,x + b .$$

Using the result of Rao and Shanbhag (1994) dealing with an extended version of integrated Cauchy functional equation, Dembińska and Wesolowski (1998) and Athar *et al.* (2003) characterized distribution by means of regression equation

$$E[X_{r+i:n} \mid X_{r:n} = x] = a\,x + b\,.$$

Characterization of distribution via linearity of regression of order statistics when gap is higher is also considered by Khan and Ali (1987) and Franco and Ruiz (1997).

Khan and Abouammoh (2000) extended the result of Khan and Abu-Salih (1989) and characterized the general form of distributions for higher order gap. Khan and Athar (2002) also characterized some continuous distributions through linearity of regression when conditioned on a pair of order statistics.

Characterization of continuous distributions by conditional variance of adjacent order statistics is first considered by Beg and Kirmani (1978). They shown that

$$V[X_{r+1:n} \mid X_{r:n} = x] = \frac{1}{a^2(n-r)^2},$$

if and only if X has exponential distribution.

Khan and Beg (1987) extended the result and proved that the conditional variance of $X^p_{r+1:n}$ given $X_{r:n} = x$ does not depend on x if and only if X Weibull distribution has.

Khan *et al.* (2009a) characterized a general class of distribution by conditional spacing of order statistics.

Xu (1998) characterized normal distribution through distribution property of order statistics whereas Alzaid and Ahsanullah (2003) characterized the exponential distribution. Further, Wesolowski and Ahsanullah (2004) characterized the power function and uniform distribution. For more results on characterization through distribution property one may refer to references given in these papers.

2. CHARACTERIZATION THROUGH MOMENTS

Let $\{n_j\}_{j=1}^{\infty}$ be a sequence of integers satisfying

$$0 < n_1 < n_2 < .... \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{1}{n_j} = \infty, \tag{2.1}$$

Then it is well known that

(i) $F(x) = 1 - \exp(-x)$, $x \geq 0$; if and only if

$E[X_{1:n_j}] = \dfrac{1}{n_j}$ for all $j \geq 1$ (Galambos and Kotz, 1978, pp. 55-57).

(ii) $F(x) = x$, $x \in (0,1)$; if and only if

$$E[X_{1:n_j}] = \frac{1}{n_j + 1}.$$

Under suitable conditions on the inverse function $F^{-1}(t) = \inf\{x : F(x) \geq t\}$, $t \in (0,1)$, Lin (1988) extend the above result by considering the relationship between $(m-1)^{th}$ and m^{th} moments of order statistics, which are stated in the following theorems.

Exponential distributions

Theorem 2.1: (G. D. Lin, 1988)

Let X be a random variable with distribution $F(x)$ and $E|X|^m < \infty$ for some constant $m \geq 1$. Let $k \geq 1$ be an integer and $\{n_j\}_{j=1}^{\infty}$ a sequence of integers satisfying (2.1). Assume that $F^{-1}(0^+) = 0$, that F^{-1} is a positive $m > 1$ is used, and further that F^{-1} is absolutely continuous on $(0,1)$. Then for given constant $\lambda > 0$, $F(x) = 1 - \exp(-x/\lambda)$, $x \geq 0$, iff

$$E[X_{k:n_j}^m] = E[X_{k-1:n_j}^m] + \frac{m\lambda}{n_j - k + 1} E[X_{k:n_j}^{m-1}], \text{ For all } n_j \geq k, \tag{2.2}$$

Omitting the first term on the right side of (2.2) when $k = 1$.

Proof: The necessary part is trivial; hence it remains to prove the sufficiency part.

On integration by parts, we write,

$$E[X_{k:n_j}^m] = k\binom{n_j}{k}\int_0^1 (F^{-1}(t))^m t^{k-1}(1-t)^{n_j-k}\,dt$$

$$= \frac{k\binom{n_j}{k}}{(n_j-k+1)}\int_0^1 (F^{-1}(t))^m t^{k-1} d[-(1-t)^{n_j-k+1}]$$

$$= \frac{k\binom{n_j}{k}}{(n_j-k+1)}\int_0^1 \{m(F^{-1}(t))^{m-1}(F^{-1}(t))'\, t^{k-1}$$

$$+(k-1)(F^{-1}(t))^m t^{k-2}\}(1-t)^{n_j-k+1}\,dt.$$

(From now on $(F^{-1}(t))' \equiv 0$ for each point t at $(F^{-1})'$ is non-defined.)

Hence, for $k=1$,

$$E[X_{1:n_j}^m] = m\int_0^1 (F^{-1}(t))^{m-1}(F^{-1}(t))'\,(1-t)^{n_j}\,dt, \tag{2.3}$$

and for $k \geq 2$,

$$E[X_{k:n_j}^m] = E[X_{k-1:n_j}^m]$$

$$+\frac{mk\binom{n_j}{k}}{(n_j-k+1)}\int_0^1 (F^{-1}(t))^{m-1}(F^{-1}(t))'\, t^{k-1}(1-t)^{n_j-k+1}\,dt, \tag{2.4}$$

It follows from (2.2) through (2.4) that for all $n_j \geq k$,

$$\int_0^1 g(t)(1-t)^{n_j-k}\,dt = 0, \tag{2.5}$$

where,

$$g(t) \equiv g_1(t) \equiv (F^{-1}(t))^{m-1}\{(F^{-1}(t))'\,(1-t)-\lambda\}\,t^{k-1},\ t \in (0,1).$$

Note that g_1 is *Lebesgue itegrable* (denoted by $g_1 \in L(0,1)$) by the assumption $E|X|^m < \infty$. Then, employing the Müntz–Szász Theorem to (2.5), we obtain $g_1(t) = 0$ almost everywhere (*a.e.*) on $(0,1)$. Therefore $(F^{-1}(t))' = \dfrac{\lambda}{(1-t)}$ *a.e.* on $(0,1)$ or equivalently,

$$F(x) = 1 - \exp(-x/\lambda),\ x \geq 0, \text{by } F^{-1}(0^+) = 0.$$

The proof is complete.

From the above proof it is seen that to apply integration by parts we need the assumption $F^{-1}(0^+) = 0$ in the only case $k = 1$. For other case $k > 1$ along with integer $m \geq 1$, excluding the assumptions $F^{-1}(0^+) = 0$ and F^{-1} is positive in Theorem 2.1, but adding the condition $P(X = 0) = 0$ if $m \neq 1$, we have the general conclusion involving both location and scale parameters: for given constant $\lambda > 0$, $F(x) = 1 - \exp(-(x-\mu)/\lambda)$ on $[\mu, \infty)$ for some constant $\mu \in R \equiv (-\infty, \infty)$, if and only if (2.2) holds.

Uniform distribution

Theorem 2.2: (G. D. Lin, 1988)

Assume that all the conditions of Theorem 2.1 hold. Then for given constant $\lambda > 0$, $F(x) = \dfrac{x}{\lambda}$, $x \in (0, \lambda)$, if and only if

$$E[X^m_{k:n_j}] = E[X^m_{k-1:n_j}] + \frac{m\lambda}{n_j + 1} E[X^{m-1}_{k:n_j+1}], \text{ for all } n_j \geq k, \tag{2.6}$$

Omitting the first term on the right side of (2.6) when $k = 1$.

Proof: The necessary part is trivial. Apropos of the sufficiency part, it follows from (2.3), (2.4) and (2.6) that (2.5) holds for all $n_j \geq k$, and for

$$g(t) \equiv g_2(t) \equiv (F^{-1}(t))^{m-1}\{(F^{-1}(t))' - \lambda\}t^{k-1}(1-t),\ t \in (0,1).$$

Again, $g_2 \in L(0,1)$ and then, employing the Müntz–Szász Theorem, $g_2(t) = 0$ *a.e.* on $(0,1)$. Therefore, $(F^{-1}(t))' = \lambda$ *a.e.* on $(0,1)$ or equivalently,

$$F(x) = \frac{x}{\lambda},\ x \in (0,\lambda), \text{ by } F^{-1}(0^+) = 0.$$

The proof is complete.

Theorem 2.3: (G. D. Lin, 1988)

Let k, l and l' be positive integers and $\{n_j\}_{j=1}^{\infty}$ a sequence of integers satisfying (2.1). Let X be a random variable $E|X|^m < \infty$ for some integer $m \ge l'$. Then for given constant $\lambda > 0$, the following statements are equivalent.

(a) $$F(x) = \left(\frac{x}{\lambda}\right)^{\frac{l'}{l}},\ x \in (0,\lambda).$$

(b) $$\alpha_{k:n_j}^{(m)} = \lambda^{l'} \alpha_{k+1:n_j+1}^{(m-l')}, \text{ for all } n_j \ge k.$$

(c) $$\alpha_{k:n_j}^{(m)} = \lambda^{l'} \{\alpha_{k+l-1:n_j+l-1}^{(m-l')} - \alpha_{k+l-1:n_j+l}^{(m-l')}\}, \text{ for all } n_j \ge k.$$

(d) $$\alpha_{k:n_j}^{(m)} - \alpha_{k:n_j+1}^{(m)} = \lambda^{l'} \alpha_{k+l+1:n_j+l+1}^{(m-l')}, \text{ for all } n_j \ge k.$$

Proof: See reference.

Theorem 2.4: (Grudzien and Szynal, 1995)

Let $\{X_n, n \ge 1\}$ be a sequence of *i.i.d.* random variables with a common distribution function F such that $E[X_1^2] < \infty$. Suppose that N is a positive Integer-valued random variable independent of $\{X_n, n \ge 1\}$ and has a probability function with $P[N=1] > 0$. Then for given $\lambda > 0$,

$$F(x) = \frac{x}{\lambda}, x \in (0,\lambda), \text{ iff}$$

$$E[X_{(N)}^2] - 2\lambda E\left[\frac{N}{N+1} X_{(N+1)}\right] + \lambda^2 E\left[\frac{N}{N+2}\right] = 0. \tag{2.7}$$

Proof: Let $F(x) = \frac{x}{\lambda}, x \in (0,\lambda)$. Then

$$E[X_{(N)}^2] = \int_0^1 (F^{-1}(t))^2 \Psi_N(t)dt = \int_0^1 t^2 \psi_N(t)dt$$

and

$$E\left[\frac{N}{N+1} X_{(N+1)}\right] = \int_0^1 (F^{-1}(t)) t \Psi_N'(t)dt = \int_0^1 t^2 \Psi_N'(t)dt.$$

Taking into account that

$$E\left[\frac{N}{N+2}\right] = \int_0^1 t^2 \psi_N'(t)dt$$

we obtain (2.7).

Now we assume that (2.7) holds true, *i.e.*

$$\int_0^1 (F^{-1}(t))^2 \Psi_N'(t)dt - 2\lambda \int_0^1 (F^{-1}(t)) t \Psi_N'(t)dt + \lambda^2 \int_0^1 t^2 \Psi_N'(t)dt = 0.$$

Then $\int_0^1 (F^{-1}(t) - \lambda t)^2 \Psi_N'(t)dt = 0$. Therefore by the assumptions we conclude that $F^{-1}(t) = \lambda t$ almost everywhere *(a.e.)* on $(0,1)$.

Logistic and Pareto distributions

Theorem 2.5: (G. D. Lin, 1988)

Let X be a random variable with distribution $F(x)$ and $E|X|^m < \infty$ for some integer $m \geq 1$. Let $k \geq 2$ be an integer and $\{n_j\}_{j=1}^{\infty}$ be a sequence of integers satisfying (2.1). Assume that $P(X = 0) = 0$ if $m \neq 1$ is used, and that F^{-1} is absolutely continuous on $(0, 1)$. Then for given constant $\lambda > 0$,

$$F(x) = \{1 + \exp(-(x - \mu)/\lambda)\}^{-1}$$

on R for some constant $\mu \in R$, if and only if for all $n_j \geq k$,

$$E[X_{k:n_j}^m] = E[X_{k-1:n_j}^m] + \frac{m\lambda n_j}{(k-1)(n_j - k + 1)} E[X_{k-1:n_j-1}^{m-1}]. \tag{2.8}$$

Proof: See reference.

Theorem 2.6: (G. D. Lin, 1988)

Let X be a random variable with distribution $F(x)$ and $E\,|\,X_{k-1:n_0}\,|^m < \infty$ for some constant $m \geq 1$ and for some positive integers $n_0 \geq k \geq 2$. Let $\{n_j\}_{j=1}^{\infty}$ be a sequence of integers satisfying (2.1). Assume that $F^{-1}(0^+) = \lambda > 0$ and F^{-1} is absolutely continuous on $(0, 1)$. Then for given constant $l \geq 1$,

$$F(x) = 1 - (x/\lambda)^{-1/l}, \quad x \geq \lambda,$$

if and only if for all $n_j \geq n_0 + l$,

$$E[X_{k:n_j}^m] = E[X_{k-1:n_j}^m] + \frac{m\lambda(n_j)!\,(n_j - k + l)!}{(n_j - l)!\,(n_j - k + 1)!} E[X_{k:n_j - l}^{m-1}] \tag{2.9}$$

Proof: See reference.

Power function distribution

Theorem 2.7: (Grudzien and Szynal, 1999)

Let $E[X_{k:n}^2] < \infty$ for some pair (k,n), then the equality

$$E[X_{k:n}^2] - \frac{2}{m}\left[\frac{n_{[k]}}{(n-m)_{[k]}} E[X_{k:n-m}] - E[X_{k:n}]\right]$$

$$+ \frac{1}{m^2}\left[\frac{n_{[k]}}{(n-2m)_{[k]}} - \frac{2n_{[k]}}{(n-m)_{[k]}} + 1\right] = 0, \tag{2.10}$$

where $n_{[k]} = n(n-1)....(n-k+1)$ and m is a negative integer holds, if and only if

$$F(x) = 1 - (1 + mx)^{-1/m}, \quad x \in (0, -1/m). \tag{2.11}$$

Proof: Let $F^{-1}(t) = \inf\{x : F(x) \geq t,\, t \in (0,1)\}$. Taking into account that

$$E[X_{k:n}^l] = \frac{n!}{(k-1)!(n-k)!} \int_0^1 (F^{-1}(t))^l\, t^{k-1}(1-t)^{n-k}\, dt, \quad l \geq 1, \tag{2.12}$$

we see that $E|X_{k:n}|<\infty$, and $E|X_{k:n-m}|<\infty$. Furthermore, when F is given by (2.11), we find that

$$E[X_{k:s}]=\frac{k}{m}\binom{s}{k}[B(k,s-m-k+1)-B(k,n-k+1)], n\le s\le n-m \quad (2.13)$$

and

$$E[X_{k:n}^2]=\frac{k}{m^2}\binom{n}{k}[B(k,n-k-2m+1)-2B(k,n-k-m+1)$$
$$+B(k,n-k+1)], \quad (2.14)$$

where $B(a,b)$ is the beta function, and so (2.10) holds true.

Conversely, assume that (2.10) holds. Applying (2.12) we see that (2.10) can be written as

$$\int_0^1\left(F^{-1}(t)-\frac{(1-t)^{-m}-1}{m}\right)^2 t^{k-1}(1-t)^{n-k}\,dt=0,$$

which implies that $F(x)$ is given by (2.11).

General Class of Distributions

Consider the general class of distribution function $F_1(x)$ as

$$F_1(x)=1-[ah(x)+b]^c;\ x\in(\alpha,\beta) \quad (2.15)$$

where $a\neq 0, b, c\neq 0$ are constants and $h(x)$ is monotonic differentiable function of x.

Let

$$I=\{(n_i)_{i\in N};\ n_1<n_2<\ldots \text{ and } \sum_{i=1}^{\infty}\frac{1}{n_i}=\infty\}$$

Then the Müntz–Szász theorem states that the sequences, $\{t^{n_i}\}_{(n_i)_{i\in I}}$ and $\{(1-t)^{n_i}\}_{(n_i)_{i\in I}}$ are complete in $L(0,1)$ (Kamps, 1991).

Theorem 2.8: (Ali and Khan, 1998b)

Let $F(\cdot)$ be the distribution function with $E|X|^l<\infty$ for some $l\ge 1$. Let the inverse function $F^{-1}(\cdot)$ be positive on [0,1] and $F^{-1}(0^+)=Q_1,\ F^{-1}(1)=P_1$.

Assume further that $\{n_i\}, i=1,2,....$ be a sequence of integers satisfying $2 \le n_1 < n_2 < ......$ and $\sum_{i=1}^{\infty} \frac{1}{n_i} = \infty$. Then for $1 \le r < s \le n_i$

$$E\{g(X_{r:n_i}, X_{s:n_i})\} - E\{g(X_{r:n_i}, X_{s-1:n_i})\}$$

$$= -\frac{n_i P_2}{(n_i - s + 1)}[E\{g(X_{r:n_i-1}, X_{s:n_i-1})\} - E\{g(X_{r:n_i-1}, X_{s-1:n_i-1})\}]$$

$$- \frac{1}{(n_i - s + 1)ca} E\{m(X_{r:n_i}, X_{s:n_i})\} \tag{2.16}$$

if and only if

$$F(x) = Q_2 - \frac{[ah(x)+b]^c}{(P-Q)}; \; x \in (Q_1, P_1) \tag{2.17}$$

where,

$$m(x,y) = [ah(y)+b]\frac{g'(x,y)}{h'(y)}, \; g'(x,y) = \frac{\partial}{\partial y} g(x,y),$$

$$h'(y) = \frac{\partial}{\partial y} h(y), \quad Q_2 = \frac{1-Q}{P-Q}, \; P_2 = \frac{1-P}{P-Q}.$$

Proof: Ali and Khan (1998a) have shown that for distribution (2.17), relation (2.16) holds and hence the "if" part.

To prove the only "if" part, we have in view of (2.15) and (2.24).

$$\frac{C_{r,s:n_i}}{(n_i - s + 1)} \iint_{0 \le u < v \le 1} g'(F^{-1}(u), F^{-1}(v)) u^{r-1} (v-u)^{s-r-1} \times (1-v)^{n_i - s + 1} du d(F^{-1}(v))$$

$$= -\frac{n_i P_2 C_{r,s:n_i-1}}{(n_i - s + 1)(n_i - s)} \iint_{0 \le u < v \le 1} g'(F^{-1}(u), F^{-1}(v)) u^{r-1} (v-u)^{s-r-1} \times (1-v)^{n_i - s} du d(F^{-1}(v))$$

$$- \frac{C_{r,s:n_i}}{(n_i - s + 1)ca} \iint_{0 \le u < v \le 1} m(F^{-1}(u), F^{-1}(v)) u^{r-1} (v-u)^{s-r-1} (1-v)^{n_i - s} du dv$$

that is $\int_0^1 (1-v)^{n_i - s} g^*(v) dv = 0$

where, $g^*(v) = [\{(1-v+P_2)(F^{-1}(v))' + \frac{ah(F^{-1}(v))+b}{cah'(F^{-1}(v))}\}$

$$\times \int_0^v g'(F^{-1}(u), F^{-1}(v)) u^{r-1}(v-u)^{s-r-1} du]$$

Under the given conditions $\{(1-v)^{n_i}\}$ is complete in $L(0,1)$(Lin, 1989 and Kamps, 1991).

Since,

$$\int_0^v g'(F^{-1}(u), F^{-1}(v)) u^{r-1}(v-u)^{s-r-1} du \neq 0.$$

Therefore, $g^*(v) = 0$, which implies that

$$\frac{cah'(F^{-1}(v))(F^{-1}(v))'}{ah(F^{-1}(v))+b} = -\frac{1}{1-v+P_2}.$$

That is,

$$\log_e[ah(F^{-1}(v))+b]^c = \log_e[1-v+P_2]+k ,$$

where k is a constant of integration which at $v=0$ gives $k=\log_e(P-Q)$. Therefore,

$$[ah(F^{-1}(v))+b]^c = (1-v+P_2)(P-Q)$$

or,

$$v = \frac{1-Q}{P-Q} - \frac{[ah(F^{-1}(v))+b]^c}{P-Q}.$$

That is,

$$F(x) = Q_2 - \frac{[ah(x)+b]^c}{P-Q};\ x \in (Q_1, P_1)$$

and hence the result.

Theorem 2.9: (Ali and Khan, 1998b)

Under the conditions of Theorem 2.8 the following statements are equivalent.

(a) $$F(x) = Q_2 - \frac{[ah(x)+b]^c}{P-Q};\quad x \in (Q_1, P_1)$$

(b) $$E\{g(X_{r:n_i})\} = E\{g(X_{r-1:n_i-1})\} - \frac{(P-Q)(n_i - r + 1)}{n_i(n_i+1)ca} E\{Z(X_{r:n_i+1})\}$$

(c) $$E\{g(X_{r:n_i})\} = E\{g(X_{r-1:n_i})\} - \frac{(P-Q)}{(n_i+1)ca} E\{Z(X_{r:n_i+1})\}$$

(d) $$E\{g(X_{r-1:n_i-1})\} = E\{g(X_{r-1:n_i-1})\} - \frac{(P-Q)(r-1)}{n_i(n_i+1)ca} E\{Z(X_{r:n_i+1})\}$$

where, $Z(x) = \dfrac{[ah(x)+b]^{1-c} g'(x)}{h'(x)}$.

Proof: See reference.

Table 2.1

Distribution Function	*a*	*b*	*c*	*h*(*x*)
Power function				
$F_1(x) = \lambda^{-p} x^p;$	$-\lambda^{-p}$	1	1	x^p
$0 \le x \le \lambda$	-1	1	1	$\lambda^{-p} x^p$
Pareto				
$F_1(x) = 1 - \lambda^p x^{-p};$	λ^p	0	1	x^{-p}
$\lambda \le x < \infty$	λ	0	p	x^{-1}
Beta of the first kind				
$F_1(x) = 1 - \left[\dfrac{(\lambda - x)}{(\lambda - \beta)}\right]^p;$	1	0	p	$(\lambda - x)/(\lambda - \beta)$
$\beta < x < \lambda$	-1	$\lambda/(\lambda - \beta)$	p	$x/(\lambda - \beta)$

Weibull				
$F_1(x) = 1 - \exp(-\theta x^p)$; $0 \le x < \infty$	1	0	1	$\exp(-\theta x^p)$
	1	0	θ	$\exp(-x^p)$
Burr type XII				
$F_1(x) = 1 - (1 + \theta x^p)^{-\lambda}$; $0 \le x < \infty$	θ	1	$-\lambda$	x^p
	1	0	$-\lambda$	$1 + \vartheta x^p$
Rectangular				
$F_1(x) = (x - \beta)/(\lambda - \beta)$; $\beta < x < \lambda$	1	0	1	$(\lambda - x)/(\lambda - \beta)$
	-1	$\lambda/(\lambda - \beta)$	1	$x/(\lambda - \beta)$
Rayleigh				
$F_1(x) = 1 - \exp(-\theta x^2)$; $0 \le x < \infty$	1	0	1	$\exp(-\theta x^2)$
	1	0	θ	$\exp(-x^2)$
Exponential				
$F_1(x) = 1 - \exp(-\theta x)$; $0 \le x < \infty$	1	0	1	$\exp(-\theta x)$
	1	0	θ	$\exp(-x)$
Lomax				
$F_1(x) = 1 - (1 + \theta x)^{-\lambda}$; $0 \le x < \infty$	θ	1	$-\lambda$	x
	1	0	$-\lambda$	$1 + \theta x$
Log logistic				
$F_1(x) = 1 - (1 + \theta x^p)^{-1}$; $0 \le x < \infty$	θ	1	-1	x^p
	1	0	-1	$1 + \theta x^p$
Cauchy				
$F_1(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x-\theta}{\lambda}\right)$; $0 < x < \infty$	$\frac{1}{\pi}$	$\frac{1}{2}$	1	$\tan^{-1}[(x-\theta)/\lambda]$

3. CHARACTERIZATION THROUGH CONDITIONAL EXPECTATION

Weibull distribution

Lemma 3.1: (Khan and Ali, 1987)

Let X be a continuous random variable having *df* $F(x)$, then $F(x)$ is unique if

$$E[X_{r+i:n}^{k} \mid X_{r:n} = x],\ 1 \le i \le n-r,\ 1 \le r \le n \text{ exists.}$$

Lemma 3.2: (Khan and Ali, 1987)

Let X be a continuous random variable having *df* $F(x)$, then $F(x)$ is unique if

$$E[X_{r:n}^{k} \mid X_{r+i:n} = x],\ 1 \le i \le n-r,\ 1 \le r \le n \text{ exists.}$$

Theorem 3.1: (Khan and Ali, 1987)

Let X be a continuous random variable having *df* $F(x)$ with $F(0)=0$ and $E(X^k) < \infty$, $k > 0$. If $F(x) < 1$ for all $x < \infty$, then $F(x) = 1 - e^{-\theta x^p}$, $x \ge 0$, $\theta > 0$, $p > 0$, if and only if for $r < n$, $1 \le i \le n-r$,

$$E[X_{r+i:n}^{p} \mid X_{r:n} = x] = x^p + \frac{1}{\theta}\sum_{l=0}^{i-1} \frac{1}{n-r-l}\ .$$

Proof: In view of Khan *et al.* (1983a), we have

$$\alpha_{r:n}^{(k)} = Q_2 \alpha_{r-1:n-1}^{(k)} - P_2\, \alpha_{r:n-1}^{(k)} + \frac{k}{np\theta} \alpha_{r:n}^{(k-p)}$$

In case of left truncation $(P=1)$, $Q_2 = 1$, $P_2 = 0$

Thus,

$$\alpha_{r:n}^{(k)} = \alpha_{r-1:n-1}^{(k)} + \frac{k}{np\theta} \alpha_{r:n}^{(k-p)}.$$

That is,

$$\alpha_{i:n-r}^{(p)} = \alpha_{i-1:n-r-1}^{(p)} + \frac{1}{(n-r)\theta}.$$

Necessary part is proved by noting that

$$\alpha_{0:n-r-i}^{(k)} = Q_1^p = x^p$$

a sufficient part follows from Lemma 3.1.

For $i=1$, the theorem was proved by Khan and Beg (1987). Also, if we put $p=1$ and $i=1$, we get the result obtained by Ferguson (1967). At $p=2$, it characterizes Rayleigh distribution.

Burr distribution

Theorem 3.2: (Khan and Ali, 1987)

Let X be a random variable having continuous *df* $F(x)$ with $F(0)=0$ and $E(X^k)<\infty$, $k>0$. If $F(x)<1$ for all $X<\infty$, then

$$F(x)=1-(1+\theta\, x^p)^{-m},\ x\geq 0,\ \theta>0,\ p>0,\ m>0$$

if and only if for $r<n$, $1\leq i\leq n-r$,

$$E[X_{r+i:n}^p \mid X_{r:n}=x]$$

$$=\begin{cases} x^p \dfrac{m(n-r)}{m(n-r)-1}+\dfrac{1}{\theta[m(n-r)-1]}, & i=1, \\[2ex] x^p \displaystyle\prod_{l=0}^{i-1}\frac{m(n-r-l)}{m(n-r-l)-1}+\frac{1}{\theta[m(n-r)-1]}[1+\sum_{j=0}^{i-2}\prod_{l=0}^{j}\frac{m(n-r-l)}{m(n-r-l)-1}], & i\geq 2 \end{cases}$$

Proof: In view of Khan and Khan (1987),

$$[1-\frac{k}{mnp}]\alpha_{r:n}^{(k)} = Q_2\,\alpha_{r-1:n-1}^{(k)} - P_2\,\alpha_{r:n-1}^{(k)} + \frac{k}{mnp\theta}\alpha_{r:n}^{(k-p)}.$$

For $k \neq mnp,\ 1 \le r \le n$, where

$$\theta Q_1^p = [(1-Q)^{-\frac{1}{m}} - 1],\ \theta P_1^p = [(1-P)^{-\frac{1}{m}} - 1],$$

$$Q_2 = \frac{1-Q}{P-Q},\ P_2 = \frac{1-p}{P-Q}.$$

Therefore, for the left truncation only, $P_2 = 0$, $Q_2 = 1$.

This gives,

$$\alpha_{i:n-r}^{(p)} = \frac{m(n-r)}{m(n-r)-1}\alpha_{i-1:n-r-1}^{(p)} + \frac{1}{\theta[m(n-r)-1]}.$$

Noting that, $\alpha_{0:n}^{(p)} = Q_1^p = x^p$, we prove the necessary part. Sufficient parts follow from Lemma 3.1.

If we put $i = 1$, we get the result of Khan and Khan (1987). For $m = 1$, it characterizes log-logistic distribution.

Pareto distribution

Theorem 3.3: (Khan and Ali, 1987)

Let X be a continuous random variable having *df* $F(x)$ with $F(a) = 0$ and $E(X^k) < \infty$, $k > 0$. If $F(x) < 1$ for all $x < \infty$, then $F(x) = 1 - a^p x^{-p}$, $x \ge a$, $a > 0, p > 0$, if and only if for $r < n$, $1 \le i \le n-r$,

$$E[X_{r+i:n}^k \mid X_{r:n} = x] = x^k \prod_{l=0}^{i-1} \frac{p(n-r-l)}{p(n-r-l)-k}$$

Proof: See reference.

Power function distribution

Theorem 3.4: (Khan and Ali, 1987)

Let X be a continuous random variable having *df* $F(x) > 0$ for $0 < x < a$ with $F(a) = 1$ and $E(X^k) < \infty$, $k > 0$,

then,

$$F(x) = a^{-p} x^p, \; 0 \le x \le a, \; a > 0, \; p > 0,$$

if and only if for $1 \le r < n$, $1 \le i \le n - r$,

$$E[X_{r:n}^k \mid X_{r+i:n} = x] = x^p \prod_{l=0}^{i-1} \frac{p(r+l)}{p(r+l)+k}.$$

Proof: See reference.

General form of distributions

$$F(x) = 1 - [ah(x) + b]^c \;, \quad x \in (\alpha, \beta).$$

Theorem 3.5: (Khan and Abu-Salih, 1989)

Let X be an absolutely continuous random variable with *df* $F(x)$ and *pdf* $f(x)$. Suppose, $F(x) < 1$ for all $x \in (\alpha, \beta)$, $F(\alpha) = 0$ and $F(\beta) = 1$.

then,

$$F(x) = 1 - [ah(x) + b]^c, \quad \text{for } x \in (\alpha, \beta)$$

if and only if for $r < n$,

$$E[h(X_{r+1:n}) \mid X_{r:n} = x] = \frac{ac(n-r)h(x) - b}{a[(n-r)c+1]} \tag{3.1}$$

where $h()$ is a monotonic, continuous and differentiable function on (α, β), $a \ne 0$, $(n-r)c + 1 \ne 0$

Proof: Note that

$$1 - F(x) = [ah(x) + b]^c = -\frac{ah(x) + b}{ach'(x)} f(x) \tag{3.2}$$

Now in view of (3.1)

$$E[h(X_{r+1:n}) \mid X_{r:n} = x]$$

$$= \frac{(n-r)}{[1-F(x)]^{n-r}} \int_x^{\beta} h(y)[1-F(y)]^{n-r-1} f(y)dy$$

Integrating by parts and noting the relation (3.2), it is easy to prove the necessary part .To prove the sufficiency part, we have

$$(n-r)\int_x^{\beta} h(y)(1-F(y))^{n-r-1} f(y)dy$$

$$= \frac{[ac(n-r)h(x)-b]}{a[c(n-r)+1]}[1-F(x)]^{n-r}$$

Differentiating both sides *w.r.t.* x and rearranging we get

$$-\frac{f(x)}{1-F(x)} = \frac{ach'(x)}{ah(x)+b}$$

which gives

$$1-F(x) = [ah(x)+b]^c.$$

Remark 3.1: At $r = n-1$,

$$a = \frac{f(k)-1}{(f(k)-1)h(\alpha)+g(k)}$$

$$b = \frac{g(k)}{(f(k)-1)h(\alpha)+g(k)}$$

and $c = \dfrac{f(k)}{1-f(k)}$

theorem reduces the result of Talwalkar (1977).

Lemma 3.3: (Khan and Abouammoh, 2000)

For any continuous and differentiable function $h()$ and $1 \le r < s \le n$.

$$E[h(X_{s:n}) | X_{r:n} = x] - E[h(X_{s-1:n}) | X_{r:n} = x]$$

$$= \binom{n-r}{s-r-1} \frac{1}{[1-F(x)]^{n-r}} \int_x^{\infty} h'(y)[F(y)-F(x)]^{s-r-1} \times [1-F(y)]^{n-s+1} dy. \tag{3.3}$$

Lemma 3.4: (Khan and Abouammoh, 2000)

For any continuous and differentiable function $h()$ and $1 \le r < s \le n$,

$$E[h(X_{r:n}) | X_{s:n} = y] - E[h(X_{r-1:s}) | X_{s:n} = y]$$

$$= \binom{s-1}{r-1} \frac{1}{[F(y)]^{s-1}} \int_{-\infty}^{y} h'(x)[F(x)]^{r-1}[F(y)-F(x)]^{s-r} dx.$$

Theorem 3.6: (Khan and Abouammoh, 2000)

Let X be an absolutely continuous random variable with *df* $F(x)$ and *pdf* $f(x)$ in the interval (α, β), where α and β may be finite or infinite. Then for continuous and differentiable function $h(x)$ of x and for $1 \le r < s \le n$

$$E[h(X_{s:n}) | X_{r:n} = x] = h(x) \prod_{j=0}^{s-r-1} \frac{c\,(n-r-j)}{c\,(n-r-j)+1}$$

$$-\frac{b}{a} \sum_{j=0}^{s-r-1} \frac{1}{c\,(n-s+1+j)} \prod_{i=0}^{j} \frac{c\,(n-s+1+i)}{c(n-s+1+i)+1} \tag{3.4}$$

if and only if,

$$F(x) = 1 - [ah(x)+b]^c, \quad \alpha \le x \le \beta, \tag{3.5}$$

where,

$a \ne 0$, $c \ne 0$, $c(n-r-j)+1 \ne 0$ for $j = 0,1,\ldots,(s-r-1)$ and $b = -ah(\beta)$.

Proof: First we have to prove that (3.5) implies (3.4), we have

$$E[h(X_{r:n})]-E[h(X_{r-1:n})]$$

$$=\binom{n}{r-1}\int_{Q_1}^{P_1} h'(x)[F(x)]^{r-1}\,[1-F(x)]^{n-r+1}dx \tag{3.6}$$

For doubly truncated distribution (3.5), we have

$$1-F(x)=-\frac{1-P}{P-Q}-\frac{ah(x)+b}{cah'(x)}f(x).$$

Expressing $[1-F(x)]^{n-r+1}$ as $[1-F(x)]^{n-r}\left\{-\frac{1-P}{1-Q}-\frac{ah(x)+b}{cah'(x)}f(x)\right\}$

and putting $Q_1=x$, $P_1=\beta$, $Q=F(x)$, $P=1$, after noting the relation between truncated and conditional distribution, we get form (3.6),

$$E[h(X_{s:n})|\,X_{r:n}=x]-E[h(X_{s-1:n})|X_{r:n}=x]$$

$$=-\binom{n}{s-1}\frac{1}{ca}\int_x^{\beta}[ah(t)+b][F(t)]^{s-1}[1-F(t)]^{n-s}f(t)dt$$

$$-\frac{1}{c(n-s+1)}\left\{E[h(X_{s:n})\,|\,X_{r:n}]+\frac{b}{a}\right\}.$$

Therefore,

$$E[h(X_{s:n})\,|\,X_{r:n}=x]=\frac{c(n-s+1)}{c(n-s+1)+1}E[h(X_{s-1:n})|\,X_{r:n}=x]$$

$$-\frac{b}{a}\frac{1}{c(n-s+1)+1}. \tag{3.7}$$

Using (3.7) recursively and noting that $E[h(X_{r:n})\,|\,X_{r:n}=x]=h(x)$, the relation (3.4) is established.

To prove that (3.4) implies (3.5), we have from (3.3) and (3.7)

$$c(n-s+1)\{E[h(X_{s:n})\,|\,X_{r:n}=x]-E[h(X_{s-1:n})\,|X_{r:n}=x]\}$$

$$=-\,E[h(X_{s:n})\,|X_{r:n}=x]-\frac{b}{a},$$

or,

$$c(n-s+1)\binom{n-r}{s-r-1}\frac{1}{[1-F(x)]^{n-r}}\int_x^\beta h'(y)\,[F(y)-F(x)]^{s-r-1}$$

$$\times[1-F(y)]^{n-s+1}\,dy$$

$$=-\frac{(n-r)!}{(s-r-1)!\,(n-s)!}\frac{1}{[1-F(x)]^{n-r}}\int_x^\beta h(y)\,[F(y)-F(x)]^{s-r-1}$$

$$\times[1-F(y)]^{n-s}f(y)\,dy-\frac{b}{a}$$

Differentiating both sides $(s-r)$ times partially $w.r.t.\ x$, after noting that

$$\frac{\partial}{\partial x}\int_{u(x)}^{v(x)}f(x,t)\,dt=\int_{u(x)}^{v(x)}\left[\frac{\partial}{\partial x}f(x,t)\right]dt+f(v,t)\frac{\partial v}{\partial x}-f(u,t)\frac{\partial u}{\partial x}$$

we get,

$$-h'(x)\,[1-F(x)]^{n-s+1}-h(x)[1-F(x)]^{n-s}f(x)$$

$$=\frac{b}{a}[1-F(x)]^{n-s}f(x)$$

or, $$-\frac{f(x)}{1-F(x)}=\frac{cah'(x)}{ah(x)+b}$$

implying

$$1-F(x)=[ah(x)+b]^c$$

and hence the theorem.

Theorem 3.7: (Khan and Athar, 2002)

For any continuous and differentiable function $h()$ and $1\le r<s\le n$,

$$E[h(X_{r+1:n})\mid X_{r:n}=x,X_{s:n}=y]$$

$$=(-1)^m P_2^m h(y)\prod_{l=1}^{m}\frac{lc}{(lc+1)}+Q_2h(x)\sum_{i=0}^{m-1}(-1)^i P_2^i\prod_{j=0}^{i}\frac{(m-j)c}{(m-j)c+1}$$

$$-\frac{b}{a}\sum_{i=0}^{m-1}(-1)^i P_2^i\frac{1}{(m-i)c}\prod_{j=0}^{i}\frac{(m-j)c}{(m-j)c+1} \tag{3.8}$$

if and only if

$$F(x) = 1-[ah(x)+b]^c, \alpha \le x \le \beta \tag{3.9}$$

where α and β are such that $F(\alpha)=0$, $F(\beta)=1$,

and $m=s-r-1$, $P_2 = \dfrac{1-P}{P-Q}$, $Q_2 = \dfrac{1-Q}{P-Q}$

$$a \neq 0, (m-i)c \neq 0, \ (m-i)c+1 \neq 0 \text{ for } i=0,1,\ldots,m-1$$

Proof: See Reference.

Theorem 3.8: (Khan *et al.*, 2009a)

Let X be an absolutely continuous random variable with *df* $F(x)$ and *pdf* $f(x)$ in the interval (α, β), where α and β may be finite or infinite, then for $1 \le m < r < s \le n$,

$$E[h(X_{s:n}) - h(X_{r:n}) \mid X_{m:n} = x] = \frac{1}{a}\sum_{j=r}^{s-1}\frac{1}{(n-j)} \tag{3.10}$$

if and only if

$$F(x) = 1-e^{-ah(x)}, \ a \neq 0 \tag{3.11}$$

Where $h(x)$ is a monotonic and differentiable function of x such that $h(x) \to 0$ as $x \to \alpha$ and $h(x)\{1-F(x)\} \to 0$ as $x \to \beta$.

Proof: First we will prove (3.11) implies (3.10). For $1 \le r < s \le n$,

$$E[h(X_{s:n}) - h(X_{s-1:n}) \mid X_{r:n} = x]$$

$$= \binom{n-r}{s-r-1}\frac{1}{[1-F(x)]^{n-r}}\int_x^\beta h'(y)[F(y)-F(x)]^{s-r-1}$$

$$\times [1-F(y)]^{n-s+1}dy$$

Therefore, for $1 \le m < r < s \le n$,

$$E[h(X_{s:n}) - h(X_{r:n}) \mid X_{m:n} = x]$$

$$= \sum_{i=0}^{s-r-1} E\left[h(X_{s-i:n}) - h(X_{s-i-1:n}) \mid X_{m:n} = x\right]$$

$$= \sum_{j=r}^{s-1} \binom{n-m}{j-m} \frac{1}{[1-F(x)]^{n-m}} \int_x^{\beta} h'(y)[F(y)-F(x)]^{j-m}$$

$$\times [1-F(y)]^{n-j} dy$$

$$= \frac{1}{a} \sum_{j=r}^{s-1} \frac{1}{(n-j)}, \text{ in view of } 1-F(x) = \frac{f(x)}{ah'(x)}$$

This proves the necessary part.

To prove the sufficiency part, let

$$c = \frac{1}{a} \sum_{j=r}^{s-1} \frac{1}{(n-j)},$$

then,

$$E[h(X_{s:n}) - h(X_{r:n}) \mid X_{m:n} = x] = c$$

implies,

$$\frac{(n-m)!}{(s-m-1)!(n-s)!} \int_x^{\beta} h(y)[F(y)-F(x)]^{s-m-1}[1-F(y)]^{n-s} f(y) dy$$

$$- \frac{(n-m)!}{(r-m-1)!(n-r)!} \int_x^{\beta} h(y)[F(y)-F(x)]^{r-m-1}[1-F(y)]^{n-r} f(y) dy$$

$$= c[1-F(x)]^{n-m} \tag{3.12}$$

Differentiating (3.12) $(r-m)$ times $w.r.t\ x$, we have,

$$= \frac{(n-r)!}{(s-r-1)!(n-s)!} \int_x^{\beta} h(y)[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(y) dy$$

$$= \{h(x)+c\}[1-F(x)]^{n-r}\ . \tag{3.13}$$

Integrating LHS of (3.13) by parts and simplifying, we get

$$\frac{(n-r)!}{(s-r-2)!(n-s+1)!}\int_x^{\beta} h(y)[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s+1} f(y)dy$$

$$+\frac{(n-r)!}{(s-r-1)!(n-s+1)!}\int_x^{\beta} h'(y)[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1} dy$$

$$=\{h(x)+c\}[1-F(x)]^{n-r} \tag{3.14}$$

From (3.13) and (3.14) it follows that

$$\frac{(n-r)!}{(s-r-1)!(n-s+1)!}\int_x^{\beta} h'(y)[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1} dy$$

$$+\{h(x)+c_1\}[1-F(x)]^{n-r}=\{h(x)+c\}[1-F(x)]^{n-r}$$

where $c_1=\frac{1}{a}\sum_{j=r}^{s-2}\frac{1}{(n-j)}$.

That is,

$$\frac{[1-F(x)]^{n-r}}{a(n-s+1)}=\frac{(n-r)!}{(s-r-1)!(n-s+1)!}\int_x^{\beta} h'(y)[F(y)-F(x)]^{s-r-1}$$

$$\times[1-F(y)]^{n-s+1} dy.$$

Differentiating $(s-r)$ times $w.r.t\ x$, we have

$$h'(x)[1-F(x)]=\frac{f(x)}{a}$$

and hence the Theorem.

4. CHARACTERIZATION THROUGH DISTRIBUTIONAL PROPERTY

Normal distribution

Theorem 4.1: (Xu, 1998)

Suppose that $n = 2$ and X_1 is symmetric about zero. Then X_1 is normally distributed if and only if

$$X_{(2)} - X_{(1)} \overset{d}{=} Y_{(2)} - Y_{(1)}, \tag{4.1}$$

where, $Y = \Gamma X$ for all $\Gamma \in O(2)$, Γ is an $n \times n$ orthogonal matrix and $O(n)$ be a group of all $n \times n$ orthogonal matrices.

Proof: For the only if part, note that

$$X_{(2)} - X_{(1)} \overset{d}{=} Y_{(2)} - Y_{(1)},$$

holds trivially if X_1 is normally distributed.

To prove the converse, note that

$$X_{(2)} - X_{(1)} = | X_2 - X_1 |.$$

It can be checked that (4.1) implies that

$$\left(\frac{| X_2 - X_1 |}{\sqrt{2}} \right)^2 \overset{d}{=} X' \Gamma' B \Gamma X, \tag{4.2}$$

where Γ is an arbitrary 2×2 orthogonal matrix, and

$$B = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

Since (4.2) is valid for any orthogonal matrix Γ, one can choose a Γ such that the right- hand side of (4.2) becomes X_1^2 because the eigen-values of B are 1 and 0. Note that X_1 is symmetric about zero by assumptions and $X_2 - X_1$ is symmetric about zero.

Thus,

$$\left(\frac{|X_2 - X_1|}{\sqrt{2}}\right)^2 \overset{d}{=} X_1^2 \text{ is equivalent to}$$

$$\frac{(X_2 - X_1)}{\sqrt{2}} \overset{d}{=} X_1. \tag{4.3}$$

Applying results in Sections 2.3 Kakosyan *et al.* (1984) we conclude that X_1 satisfying (4.3) is normally distributed.

The following Lemma taken from Kotlarski (1967) is needed in the proof of Theorem 4.2.

Lemma 4.1: (Xu, 1998)

Let X_1, X_2, X_3 be three independent positive random variables and let (Y_1, Y_2) be the pair of quotients given by

$$Y_1 = \frac{X_1}{X_3}, \quad Y_2 = \frac{X_2}{X_3}.$$

The necessary and sufficient condition for X_k to be gamma distributed with parameters α_k and λ (λ – same for $k = 1,2,3$) is that the joint distribution of (Y_1, Y_2) is the bivariate beta distribution of the second kind given by the density

$$g(y_1, y_2) = \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \frac{y_1^{\alpha_1 - 1} y_2^{\alpha_2 - 1}}{(1 + y_1 + y_2)^{\alpha_1 + \alpha_2 + \alpha_3}}, \; y_1 > 0, y_2 > 0;$$

$$= 0, \text{ otherwise.}$$

Theorem 4.2: (Xu, 1998)

Suppose that $n = 3$ and X_1 has a continuous density function f satisfying $\int_0^\infty y^{-2/3} f(y)dy < \infty$. Then X_1 is normally distributed if and only if

$$X_{(3)} - X_{(1)} \overset{d}{=} Y_{(3)} - Y_{(1)}, \tag{4.4}$$

where $Y = \Gamma X$ for all $\Gamma \in O(3)$.

Proof: When X_1 is normally distributed, (4.4) holds trivially. To prove the converse, dividing both $P(|X_{(3)}-X_{(1)}|\le t)$ and $P(|Y_{(3)}-Y_{(1)}|\le t)$ by t^2 and passing to the limit as $t\to 0$ yields

$$\int_{-\infty}^{\infty} f^3(x)dx = \int_{-\infty}^{\infty} f(ax)f(bx)f(cx)dx \tag{4.5}$$

for arbitrary real numbers a, b and c satisfying $a^2+b^2+c^2=3$. This implies that

$$\int_{-\infty}^{\infty}[f(x)-f(-x)]^2 f(x)dx \equiv 0.$$

Thus $f(x)=f(-x), -\infty<x<\infty$ follows from continuity of f immediately. Note that (4.5) is equivalent to

$$\int_{-\infty}^{\infty} f(ax)f(bx)f(cx)dx = f^2(0),\ a^2+b^2+c^2=1. \tag{4.6}$$

Let

$$\overline{G}(x)=\int_x^{\infty}\frac{z^{-5/6}f(\sqrt{z})dz}{A},\ x\ge 0,$$

where $A=2\int_0^{\infty} y^{-2/3}f(y)dy$. Then (4.6) can be expressed in terms of function $\overline{G}$ by

$$\int_0^{\infty} x^2\overline{G}'(\alpha x)\overline{G}'(\beta x)d[1-\overline{G}(x)]=\frac{h^2(0)}{A^3}\frac{\alpha^{-5/6}\beta^{-5/6}}{(1+\alpha+\beta)^{1/2}}. \tag{4.7}$$

It is easy to check that that function $\overline{G}(x)$ is a survival function on $[0,\infty)$. Integrating both sides of (4.7) with respect to (α,β) on $[\theta_1,\infty)\times[\theta_2,\infty)$ yields

$$\int_0^{\infty}\overline{G}(\theta_1 x)\overline{G}(\theta_2 x)d[1-\overline{G}(x)]=\int_{\theta_1}^{\infty}\int_{\theta_2}^{\infty}\frac{h^2(0)}{A^3}\frac{\alpha^{-5/6}\beta^{-5/6}}{(1+\alpha+\beta)^{1/2}}d\alpha d\beta \tag{4.8}$$

where $\theta_1\ge 0, \theta_2\ge 0$. Note that the left-hand side of (4.8) is equal to

$$P\left[\frac{Z_1}{Z_3} > \theta_1, \frac{Z_2}{Z_3} > \theta_2\right],$$

where random variables Z_1, Z_2 and Z_3 are independent and have the same survival function $\overline{G}(x)$. Putting $\theta_1 = \theta_2 = 0$ into (4.8) we conclude that

$$\frac{h^2(0)}{A^3} = \frac{\Gamma(1/2)}{\Gamma^3(1/6)}.$$

Since

$$P\left[\frac{Z_1}{Z_3} > 0, \frac{Z_2}{Z_3} > 0\right] = 1$$

and

$$\int_0^\infty \int_0^\infty \frac{\alpha^{-5/6}\beta^{-5/6}}{(1+\alpha+\beta)^{1/2}} d\alpha\, d\beta = \frac{\Gamma^3(1/6)}{\Gamma(1/2)}.$$

Thus, using (4.7) and Lemma 4.1 with $\alpha_1 = \alpha_2 = \alpha_3 = 1/6$ we conclude that Z_1 has a gamma distribution with parameters $(1/6, \lambda)$.

That is,

$$\frac{f(\sqrt{z})z^{-5/6}}{A} = \left[\frac{\lambda^{1/6}}{\Gamma(1/6)}\right] z^{-5/6} e^{-\lambda z} \text{ for some } \lambda > 0$$

which implies that

$$f(z) = [\sqrt{\lambda/\pi} f^2(0)]^{1/3} e^{-\lambda z^2}, \quad z \geq 0. \tag{4.9}$$

Using the symmetric property $f(z) = f(-z)$ in (4.9) we conclude that f is the density of a normal distribution with mean zero and variance $1/(2\pi f^2(0))$.

Exponential distribution

Theorem 4.3: (Alzaid and Ahsanullah, 2003)

Let X be a non-negative random variable having an absolutely continuous (with respect to Lebesgue measure) distribution function $F(x)$. Then the following two properties are equivalent.

(a) X has an exponential distribution with $F(x) = 1 - e^{-x}$, $x > 0$.

(b) For some fixed $r > 1$,

$$X_{r:n} \underline{\underline{d}} X_{r-1:n} + \frac{W}{(n-r+1)}$$

where W is distributed as exponential.

Proof: It is well known that (a) implies (b). To show (b) implies (a), we have

$$F_{r:n}(x) = \int_0^x P[W < (n-r+1)(x-y)] f_{r-1:n}(y)dy$$

$$= \int_0^x (1 - e^{-(n-r+1)(x-y)}) f_{r-1:n}(y)dy$$

$$= F_{r-1:n}(x) - \int_0^x (1 - e^{-(n-r+1)(x-y)}) f_{r-1:n}(y)dy \tag{4.10}$$

Thus

$$e^{(n-r+1)x}[F_{r-1:n}(x) - F_{r:n}(x)] = \int_0^x (e^{(n-r+1)y}) f_{r-1:n}(y)dy \tag{4.11}$$

It can be shown that

$$[F_{r-1:n}(x) - F_{r:n}(x)] = \frac{f_{r-1:n}(x)F(x)}{(r-1)f(x)} \tag{4.12}$$

Using the relation

$$\frac{f_{r-1:n}(x)}{f(x)} = (r-1)c_{r-1:n}(F(x))^{r-2}(1-F(x))^{n-r+1},$$

we obtain on simplification

$$[e^{(n-r+1)x}c_{r-1,n}(F(x))^{r-1}(1-F(x))^{n-r+1}]$$

$$= \int_0^x (e^{(n-r+1)y}) f_{r-1:n}(y)dy \tag{4.13}$$

Differentiating both sides of equation (4.13), we get

$$(n-r+1)e^{(n-r+1)x}[c_{r-1,n}(F(x))^{r-1}(1-F(x))^{n-r+1}$$

$$+(r-1)(F(x))^{r-2}c_{r-1,n}(1-F(x))^{n-r+1}f(x)$$

$$-(n-r+1)(F(x))^{r-1}c_{r-1,n}(1-F(x))^{n-r+2}f(x)]$$

$$= e^{(n-r+1)x}f_{r-1:n}(x) \tag{4.14}$$

Substituting

$$f_{r-1:n}(x) = (r-1)c_{r-1,n}(F(x))^{r-2}(1-F(x))^{n-r+1}f(x),$$

we have on simplification

$$\frac{f(x)}{1-F(x)} = 1 \tag{4.15}$$

The solution of equation (4.15) is

$$F(x) = 1 - ce^{-x} \tag{4.16}$$

Since $F(x)$ is a distribution function with $F(0)=0$, we must have $c=1$.

$$F(x) = 1 - e^{-x}, \ -\infty < x < \infty. \tag{4.17}$$

Power distribution

Theorem 4.4: (Wesolowski and Ahsanullah, 2004)

Let $U \underline{\underline{d}}\ pow(1,\alpha)$ for some $\alpha > 0$ be independent of $X_1, X_2, ..., X_n$, which are positive *i.i.d.* random variables. If

$$X_{r:n} \underline{\underline{d}}\ X_{r:n-1}U \tag{4.18}$$

for an arbitrary but fixed $r \in \{1,...,n-1\}$ then $a > 0$ exists such that

$$X_1 \underline{\underline{d}}\ pow(a, \alpha / n).$$

Proof: By (4.18), for any $x > 0$,

$$F_{r:n}(x) = \int_0^1 F_{r:n-1}\left(\frac{x}{u}\right)\alpha u^{\alpha-1} du.$$

Denote

$$a = \sup\{x > 0 : F(x) < 1\} \le \infty.$$

Then (4.18) implies

$$\inf\{x > 0 : F(x) > 0\} = 0.$$

Substituting $t = \dfrac{x}{u}$ in the above integral we obtain

$$F_{r:n}(x) = \alpha \int_0^{x/a} u^{\alpha-1} du + \alpha \int_{x/a}^1 F_{r:n-1}\left(\frac{x}{u}\right) u^{\alpha-1} du$$

$$= \left(\frac{x}{a}\right)^{\alpha} + \alpha x^{\alpha} \int_x^a F_{r:n-1}(t) t^{-\alpha-1} dt,$$

where it is understood that $(x/a)^{\alpha} = 0$ in the case $a = \infty$. observe that the last expression is differentiable in x. Hence it follows that $F_{r:n}$ is also differentiable. Consequently the density f of X_1 and the densities of all order statistics exist. Upon differentiation we get

$$f_{r:n}(x) = \frac{\alpha x^{\alpha-1}}{a^{\alpha}} + \alpha^2 x^{\alpha-1} \int_x^a F_{r:n-1}(t) t^{-\alpha-1} dt - \alpha x^{-1} F_{r:n-1}(x).$$

Substituting for integral from the previous equation we arrive at

$$x f_{r:n}(x) = \alpha[F_{r:n}(x) - F_{r:n-1}(x)], \qquad (4.19)$$

holding for any $x \in (0,a)$. For any $x \in (0,a)$, the difference $F_{r:n}(x) - F_{r:n-1}(x)$ is non-zero, since $\inf\{x > 0 : F(x) > 0\} = 0$.

Consequently the density f is always positive in $(0,a)$ and (4.19) leads to

$$\frac{F'(x)}{F(x)} = \frac{\alpha}{nx},$$

where $F(x) = Kx^{\alpha/n}$ for any $x \in (0,a)$, which yields $a < \infty$ and $K = a^{-\alpha/n}$.

Remark 4.1: For U in the above result we can take $U_{n:n}$, where $U_1, \ldots, U_n$ is random sample from the uniform (or, more generally, *pow* $(1,\alpha)$ distribution. Then (4.18) implies that X_1 is uniform on $(0,a)$ (or more generally, *pow* (a,α)).

Theorem 4.5: (Wesolowski and Ahsanullah, 2004)

Let $U \underline{\underline{d}}\ pow(1,\alpha)$ for some $\alpha > 0$ be independent of $X_1, X_2, \ldots, X_n$, which are positive, *i.i.d.* random variables. If

$$X_{r:n} \underline{\underline{d}}\ X_{r+1:n} U \tag{4.20}$$

for an arbitrary but fixed $r \in \{1, \ldots, n-1\}$ then $a > 0$ exists such that

$$X_1 \underline{\underline{d}}\ pow\,(a, \alpha/r).$$

Proof: As in the proof of Theorem 4.4 denote $a = \sup\{x > 0 : F(x) < 1\} \le \infty$ and observe that $\inf\{x > 0 : F(x) > 0\} = 0$. By (4.20) we write at

$$F_{r:n}(x) = \int_0^1 F_{r+1:n}\left(\frac{x}{u}\right)\alpha u^{\alpha-1} du = \left(\frac{x}{a}\right)^{\alpha} + \alpha x^{\alpha} \int_x^a F_{r+1:n}(t) t^{-\alpha-1} dt$$

for any $x \in (0,a)$. again the form of the above equation implies existence of densities. Then

$$x f_{r:n}(x) = \alpha[F_{r:n}(x) - F_{r+1:n}(x)],\ x \in (0,a). \tag{4.21}$$

Since $F_{r:n}(x) - F_{r+1:n}(x)$ and, consequently, $f_{r:n}(x)$ are non-zero, and thus $f(x)$ is also non-zero on $(0,a)$. Thus from (4.21) for any $x \in (0,a)$, we get

$$\frac{F'(x)}{F(x)} = \frac{\alpha}{rx},$$

leading to $F(x) = Kx^{\alpha/r}$ in $(0,a)$, implying $a < \infty$ and $K = a^{-\alpha/r}$ and the hypothesis.

Remark 4.2: In Theorem 4.5, taking $U_{r:r}$ for U as a special case where $U_1, U_2, \ldots, U_r$ is a random sample from the $U(0,1)$ (or $pow\,(1,\alpha)$) distribution, results in the $U(0,a)$ (or $pow\,(a,\alpha)$) distribution for X_1.

Theorem 4.6: (Wesolowski and Ahsanullah, 2004)

Let U_i be an random variable with the power distribution $pow(1,\alpha_i)$, $i = 1,2$, and let U_1, U_2 be independent of $X_1, X_2, \ldots, X_n$, which are positive absolutely continuous *i.i.d. rvs* such that $\inf\{x > 0 : F(x) > 0\} = 0$. Assume that

$$X_{r:n-1}U_1 \underline{\underline{d}}\, X_{r+1:n}U_2 \tag{4.22}$$

for an arbitrary but fixed $r \in \{1, \ldots, n-1\}$. Then $\dfrac{\alpha_1}{n} = \dfrac{\alpha_2}{r} = \alpha$, say, and

there exists $a > 0$ such that $X_1 \underline{\underline{d}}\, pow\,(a,\alpha)$.

Proof: Denote, as earlier, $a = \sup\{x > 0 : F(x) < 1\} \le \infty$. Then (4.22) is equivalent to

$$\int_0^1 F_{r:n-1}\left(\frac{x}{u}\right)\alpha_1 u^{\alpha_1 - 1} du = \int_0^1 F_{r+1:n}\left(\frac{x}{u}\right)\alpha_2 u^{\alpha_2 - 1} du, \quad x \in (0,a),$$

and with similarity to the previous section, it follows that for any $x \in (0,a)$

$$\left(\frac{x}{a}\right)^{\alpha_1} + \alpha_1 x^{\alpha_1} \int_x^a F_{r:n-1}(t)\, t^{-\alpha_1 - 1} dt$$

$$= \left(\frac{x}{a}\right)^{\alpha_2} + \alpha_2 x^{\alpha_2} \int_x^a F_{r+1:n}(t)\, t^{-\alpha_2 - 1} dt$$

Consider first the case $\alpha_1 = \alpha_2$. Then the above equation implies

$$F_{r:n-1}(x) = F_{r+1:n}(x), \text{ for all } x \in (0,a).$$

But this is impossible. So, $\alpha_1 \ne \alpha_2$.

Taking the derivative with respect to x, we get

$$x^{-\alpha_2}[\alpha_2 F_{r+1:n}(x) - \alpha_1 F_{r:n-1}(x)]$$

$$= (\alpha_2 - \alpha_1)\left(a^{-\alpha_2} + \alpha_2 \int_x^a F_{r+1:n}(t) t^{-\alpha_2 - 1} dt\right)$$

Differentiating again we obtain, after some simple algebra,

$$x[\alpha_2 f_{r+1:n}(x) - \alpha_1 f_{r:n-1}(x)]$$

$$= \alpha_1 \alpha_2 [F_{r+1:n}(x) - F_{r:n-1}(x)], \quad x \in (0,a).$$

The expressions on both sides of this equation are non-zero. But, the expression on the left-hand side equals

$$x\frac{n-r}{1-F(x)}\left(\frac{\alpha_2 F(x)}{r} - \frac{\alpha_1}{n}\right) f_{r:n}(x),$$

so $f_{r:n}$ is non-zero, and hence f also is non-zero. Then we arrive at

$$\frac{f(x)}{F(x)[1-F(x)]}[c_1 - c_2 F(x)] = \frac{c_1 c_2}{x}, \quad x \in (0,a),$$

where $c_1 = \frac{\alpha_1}{n}$ and $c_2 = \frac{\alpha_2}{r}$. the above equation implies $c_1 \geq c_2$. Also it can be rewritten as a simple differential equation in $(0,a)$,

$$c_1 \frac{F'(x)}{F(x)} + (c_1 - c_2)\frac{F'(x)}{1-F(x)} = \frac{c_1 c_2}{x}.$$

Hence it follows that

$$F(x)^{c_1}[1-F(x)]^{c_1 - c_2} = K x^{c_1 c_2} \tag{4.23}$$

for any $x \in (0,a)$, where K is non-zero real constant. This implies $a < \infty$. observe now that for $x \to a$ the left-hand side of (4.23) tends to zero if $c_1 > c_2$ while the right-hand side tends to $K a^{c_1 c_2} > 0$. Hence $c_1 = c_2 = \alpha$, say, and (4.23) then has the form $F(x) = K_1 x^{\alpha}$, $x \in (0,a)$, with $K_1 = a^{-\alpha}$.

Remark 4.3: Similarly, there is an interesting case in Theorem 4.6 if we take $U_{n:n}$ and $r \in \{1,\ldots,n-1\}$ for U_1 and U_2, respectively, where $U_1, U_2, \ldots, U_n$ is a sample from the $U(0,1)$ (or $pow\,(1,\alpha)$) distribution. Then it follows that $X_1 \underline{\underline{d}}\, U(0,a)$(or $pow\,(1,\alpha)$).

CHAPTER III

CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS THROUGH RECORD VALUES

1. INTRODUCTION

Characterizations of distributions through record values have been extensively studied by many authors in the literature.

Kirmani and Beg (1984) and Lin (1987, 1988) characterized probability distributions based on the moments of record values. Too and Lin (1989) characterized exponential distribution via moments of record values whereas Grudzien and Szynal (1997, 1999) characterized uniform, exponential and power function distribution in terms of $k-th$ record values. Characterization of some particular distributions using relations of expected values of records are given by Nagraja (1977,1988), Ahsanullah (1982, 1990 & 1991), Gupta (1984), Lin (1988), Roy (1990), Ahsanullah and Kirmani (1991) and Balakrishnan *et al.* (1992) and others.

Characterization of distribution through conditional expectation of record values have been considered by Nagraja (1988), Franco and Ruiz (1996, 1997), López-Blázquez and Moreno-Rebollo (1997), Ahsanullah and Wesolowski (1998), Dembińska and Wesolowski (2000) and Athar *et al*. (2003), and Wu (2004).

Nagaraja (1977) characterized continuous distributions by linearity of regression of record values, through relation

$$E\,[X_{u(r+1)} \mid X_{u(r)} = x] = ax + b.$$

Further, Nagaraja (1988) also characterized distributions by means of

$$E\,[X_{u(r)} \mid X_{u(r+1)} = x] = ax + b.$$

Wesolowski and Ahsanullah (1997) extended the result of Nagaraja (1977) and characterized the distributions for double order gap.

López-Blázquez and Moreno-Rebollo (1997), Dembińska and Wesolowski (2000) and Athar *et al*. (2003) characterized distributions for higher order gap by the means of the relation

$$E\,[X_{u(r+i)} \mid X_{u(r)} = x] = ax + b\,.$$

Lee (2001) characterized exponential distribution by conditional expectation of record values, whereas Lee (2003) characterized Pareto distribution.

Gupta and Ahsanullah (2004) characterized distribution through conditional expectation of record values through

$$E\,[\xi\{X_{u(r+2)}\} \mid X_{u(r)} = x] = g(x),$$

where $g(x)$ may be non linear but differentiable *w.r.t.* x.

Further, Bairamov *et al.* (2005) characterized exponential type distribution via regression on pairs of record values, where regression may not be linear.

Ahsanullah (1991) and Ahsanullah (2010) characterized exponential distribution through distribution properties of record value whereas Alzaid and Ahsanullah (2003) characterized Gumbel distribution and Chang (2007) characterized Pareto distribution.

2. CHARACTERIZATION THROUGH MOMENTS

Uniform distribution

Theorem 2.1: (Grudzien and Szynal, 1997)

Let $\{X_n, n \geq 1\}$ be a sequence of *i.i.d.* random variables with a common distribution function $F(x)$, and assume $E\,|\min(X_1, \cdots, X_k)|^{2p} < \infty$ for a fixed $k \geq 1$ and some $p > 1$. Suppose that N is a positive integer-valued random variable independent of $\{X_n, n \geq 1\}$ such that

$$E\left[\frac{\Gamma^{1/q}((N-1)q+1)}{(N-1)!}\right] < \infty, \tag{2.1}$$

where, $q = \dfrac{p}{(p-1)}$. Then $F(x) = x^{1/m}$, $x \in (0,1)$, for a positive integer m, if and only if

$$E[Y_N^{(k)}]^2 - 2\sum_{l=0}^{m}\binom{m}{l}(-1)^l E\left(\left(\frac{k}{k+l}\right)^N Y_N^{(k+l)}\right)$$

$$+\sum_{l=0}^{2m}\binom{2m}{l}(-1)^l E\left(\frac{k}{k+l}\right)^N = 0 \tag{2.2}$$

Let

$$F^{-1}(t) = \inf\{x : F(x) \geq t\},\ t \in (0,1).$$

It is known that

$$E[Y_n^{(k)}]^l = \frac{k^n}{(n-1)!}\int_0^1 [F^{-1}(t)]^l (1-t)^{k-1}[-\log(1-t)]^{n-1}\,dt. \tag{2.3}$$

First consider non-random N (as in Theorem 2.2 below). Observe that the left hand side of (2.2) in that case equals

$$a(n) = \frac{k^n}{(n-1)!}\int_0^1 [F^{-1}(t) - t^m]^2[-\log(1-t)]^{n-1}(1-t)^{k-1}\,dt.$$

Since,

$$\sum_{l=0}^{m}\binom{m}{l}(-1)^l E\left(\frac{k}{k+l}\right)^n Y_N^{(k+l)}$$

$$= \frac{k^n}{(n-1)!}\int_0^1 F^{-1}(t)t^m[-\log(1-t)]^{n-1}(1-t)^{k-1}\,dt$$

and

$$\sum_{l=0}^{2m}\binom{2m}{l}(-1)^l E\left(\frac{k}{k+l}\right)^n = \frac{k^n}{(n-1)!}\int_0^1 t^{2m}[-\log(1-t)]^{n-1}(1-t)^{k-1}\,dt.$$

Hence, by independence of N and X's, (2.2) is equivalent to

$$\sum_{n=1}^{\infty} a(n)P[N = n] = 0,$$

which is equivalent to

$$F(x) = x^{1/m},\ x \in (0,1).$$

For the case $m = 1$, Theorem 2.1 gives the following characterization of the uniform distribution. We have $F(x) = x,\ x \in (0,1)$,

if and only if

$$E[Y_N^{(k)}]^2 - 2\left[E[Y_N^{(k)}] - E\left(\frac{k}{k+1}\right)^n Y_N^{(k+1)}\right]$$

$$+1 - 2E\left(\frac{k}{K+1}\right)^N + E\left(\frac{k}{K+2}\right)^N = 0.$$

In the case $k = 1$, we get from Theorem 2.1 the following characterizing conditions of the uniform distribution by moments of record values.

Theorem 2.2: (Grudzien and Szynal, 1997)

Let $\{X_n, n \geq 1\}$ be a sequence of *i.i.d.* random variables with a common distribution function $F(x)$, such that $E|X_1|^{2p} < \infty$ for some fixed $p > 1$. suppose that N is a positive integer-valued random variable independent of $\{X_n, n \geq 1\}$ such that (2.1) is satisfied. Then $F(x) = x^{1/m},\ x \in (0,1)$, for some positive integer $m \geq 1$, if and only if

$$E[X_{L(N)}^2] - 2\sum_{l=0}^{m}\binom{m}{l}(-1)^l E\left(\frac{1}{1+l}\right)^N X_{L(N):L(N)+l}$$

$$+\sum_{l=0}^{2m}\binom{2m}{l}(-1)^l E\left(\frac{1}{1+l}\right)^N = 0.$$

We note that $F(x) = x,\ x \in (0,1)$, if and only if

$$E[X_{L(N)}^2] - 2E[X_{L(N)}] + E2^{-N+1} X_{L(N):L(N)+l}$$

$$-E2^{-N+1} + E3^{-N} + 1 = 0$$

Characterization conditions of the uniform distribution via moments of the *kth* record values $Y_n^{(k)}$, *i.e.* when $P[N=n]=1$, are given in the following Theorem 2.3.

Theorem 2.3: (Grudzien and Szynal, 1997)

Let $\{X_n, n \ge 1\}$ be a sequence of *i.i.d.* random variables with a common distribution function $F(x)$, such that $E|\min(X_1, \cdots, X_k)|^{2p} < \infty$ for a fixed $k \ge 1$ and some $p > 1$. Then $F(x) = x^{1/m}$, $x \in (0,1)$, for a positive integer m, if and only if

$$E[Y_n^{(k)}]^2 - 2\sum_{l=0}^{m}\binom{m}{l}(-1)^l\left(\frac{k}{k+l}\right)^n E[Y_n^{(k+l)}]$$

$$+\sum_{l=0}^{2m}\binom{2m}{l}(-1)^l E\left(\frac{k}{k+l}\right)^n = 0 \tag{2.4}$$

In the case $n=1$, the condition

$$E[X_{1:k}^2] - 2\binom{k+m}{l}^{-1} E[X_{1+m:k+m}] + \binom{k+2m}{l}^{-1} = 0$$

characterizes $F(x) = x^{1/m}$, $x \in (0,1)$, when $E[X_{1:k}^2] < \infty$.

Weibull distribution

Theorem 2.4: (Grudzien and Szynal, 1997)

Let $\{X_n, n \ge 1\}$ be a sequence of *i.i.d.* random variables with a common distribution function $F(x)$, and assume $E|\min(X_1, \cdots, X_k)|^{2p} < \infty$ for a fixed $k \ge 1$ and some $p > 1$. Suppose that N is a positive integer-valued random variable independent of $\{X_n, n \ge 1\}$. Then

$$F(x) = 1 - \exp(-x^{1/m}),\ x > 0,$$

for a positive integer m, if and only if

$$E[Y_N^{(k)}]^2 - 2k^{-m}E\left[\frac{(N+m-1)!}{(N-1)!}Y_{N+m}^{(k)}\right]$$

$$+k^{-2m}E\left[\frac{(N+2m-1)!}{(N-1)!}\right] = 0\ , \qquad (2.5)$$

provided that $E[N^{2m}] < \infty$, $m \geq 1$.

In the case $m = 1$ the condition

$$E[Y_N^{(k)}]^2 - 2k^{-1}E[NY_{N+1}^{(k)}] + k^{-2m}E[N(N+1)] = 0$$

characterizes $F(x) = 1 - \exp(-x)$, $x > 0$, when $E[N^2] < \infty$.

Proof: First consider non-random N. Then the left hand side of (2.5) equals

$$b(n) = \frac{k^n}{(n-1)!}\int_0^1 \left[F^{-1}(t) - \left(\log\frac{1}{1-t}\right)^m\right]^2 [-\log(1-t)]^{n-1}(1-t)^{k-1}dt$$

as

$$E\left[\frac{(n+m-1)!}{(n-1)!k^m}Y_{n+m}^{(k)}\right] = \int_0^1\left(\log\frac{1}{1-t}\right)^{2m+n-1}(1-t)^{k-1}dt$$

$$= \frac{(n+2m-1)!}{(n-1)!k^{2m}}.$$

Hence by independence of N and X's, (2.5) is equivalent to

$$\sum_{n=1}^{\infty} b(n)P[N = n] = 0,$$

which is equivalent to

$$F(x) = 1 - \exp(-x),\ x > 0.$$

Theorem 2.5: (Lin, 1988)

Let X be a random variable (rv) with continuous distribution $F(x)$ and $E|X|^{m+\varepsilon} < \infty$ for some integer $m \ge 1$ and for some $\varepsilon > 0$. Then for given constant $\lambda > 0$ and positive integers n_0, l and $l' = m$,

$$F(x) = 1 - \exp\left[-\left(\frac{x}{\lambda}\right)^{l'/l}\right],\ x \ge 0,$$

if and only if

$$E[X_{L(n)}^{m}] = \lambda^{l'} \frac{(n+l)!}{n!} E[X_{L(n+l)}^{m-l'}] \text{ for all } n \ge n_0. \tag{2.6}$$

Proof: See reference.

Power function distribution

Theorem 2.6: (Grudzien and Szynal, 1999)

Let $\{X_n, n \ge 1\}$ be a sequence of $i.i.d.$ random variables with a common distribution function $F(x)$ such that $E|\min(X_1,\cdots,X_k)|^{2p} < \infty$ for a fixed $k \ge 1$ and some $p > 1$. Then $F(x)$ is given by

$$F(x) = 1 - (1+mx)^{-1/m},\ \ x \in (0, -1/m) \tag{2.7}$$

if and only if

$$E[Y_n^{(k)}]^2 - \frac{2}{m}\left[\left(\frac{k}{k-m}\right)^n E[Y_n^{(k-m)}] - E[Y_n^{(k)}]\right]$$

$$+\frac{1}{m^2}\left[1 - 2\left(\frac{k}{k-m}\right)^n + \left(\frac{k}{k-2m}\right)^n\right] = 0 \text{ for } n = 1,2,\ldots \tag{2.8}$$

Proof: Suppose that $F(x)$ is given by (2.7). Then we have

$$E[Y_n^{(k)}] = \frac{k^n}{(n-1)!}\int_0^1 F^{-1}(t)[-\log(1-t)]^{n-1}(1-t)^{k-1}dt$$

$$= \frac{1}{m}\left[\left(\frac{k}{k-m}\right)^m - 1\right] \tag{2.9}$$

and

$$E[Y_n^{(k)}]^2 = \frac{k^n}{(n-1)!}\int_0^1 [F^{-1}(t)]^2[-\log(1-t)]^{n-1}(1-t)^{k-1}dt$$

$$= \frac{k^n}{(n-1)!}\int_0^1 [(1-t)^{-m} - 1]^2[-\log(1-t)]^{n-1}(1-t)^{k-1}dt$$

$$= \frac{k^n}{(n-1)!m^2}\left[\frac{\Gamma(n)}{(k-2m)^n} - \frac{2\Gamma(n)}{(k-m)^n} + \frac{1}{k^n}\Gamma(n)\right]$$

$$= \frac{1}{m^2}\left[\left(\frac{k}{k-2m}\right)^n + \left(\frac{k}{k-m}\right)^n + 1\right] \tag{2.10}$$

$$\left(\frac{k}{k-m}\right)^n E[Y_n^{(k-m)}] = \frac{k^2}{m}\left[\frac{1}{(k-2m)^n} + \frac{1}{(k-m)^n}\right] \tag{2.11}$$

which establishes (2.8).

conversely, assuming that (2.8) is satisfied, then we see that

$$\int_0^1 \left[[F^{-1}(t)] - \frac{(1-t)^{-m} - 1}{m}\right]^2 [-\log(1-t)]^{n-1}(1-t)^{k-1}dt = 0.$$

Since the sequence $\{(-\log(1-t))^n, n \geq 1\}$, is complete in $L(0,1)$, we conclude that $F(x)$ is of the form (2.7).

General form of distributions

Theorem 2.7: (Lin, 1988)

For given positive integer k, n_0 and m, the sequence

$$\{(n+k)!E[X_{L(n+k)}^m]-n!E[X_{L(n)}^m]\}_{n=n_0}^{\infty}$$

characterizes $F(\cdot)$ provided $E|X|^{m+\varepsilon}<\infty$ for some $\varepsilon>0$.

Proof: Note that the assumption $E|X|^{m+\varepsilon}<\infty$ implies $E|X_{L(n)}^m|^{m+\varepsilon}<\infty$ for all $n\geq 0$ (Lin, 1987, Lemma 1). Suppose Y is a random variable with continuous distribution G, $E|Y|^{m+\varepsilon}<\infty$ and for all $n\geq n_0$,

$$(n+k)!E[X_{L(n+k)}^m]-n!E[X_{L(n)}^m]$$

$$=(n+k)!E[Y_{L(n+k)}^m]-n!E[y_{L(n)}^m] \tag{2.12}$$

Then we want to prove $F=G$. It is seen that (2.12) is equivalent to

$$\int_0^1[h_1(t)-h_2(t)][-\log(1-t)]^{n-n_0}\,dt=0 \text{ for all } n\geq n_0, \tag{2.13}$$

where,

$$h_i(t)=[H_i(t)]^m[\{-\log(1-t)\}^k-1][-\log(1-t)]^{n_0},\ i=1,2$$

$$H_1(t)=F^{-1}(t) \text{ and } H_2(t)=G^{-1}(t),\ t\in(0,1).$$

Furthermore, by the finiteness of $E|X|^{m+\varepsilon}$ and $E|Y|^{m+\varepsilon}$, $h_i\in L_p(0,1)$, $i=1,2$, where $p\equiv(m+\frac{1}{2}\varepsilon)/m>1$. Since the sequence of functions $\{(-\log(1-t))^n\}_{n=0}^{\infty}$ is complete in $L(0,1)$ (Lin, 1987, Lemma 3), $h_1(t)=h_2(t)$ almost everywhere (*a.e.*) on $(0,1)$ by (2.13).

Therefore,

$$[F^{-1}(t)]^m=[G^{-1}(t)]^m \text{ almost everywhere on } (0,1).$$

Note that $F^{-1}(\cdot)$ is strictly increasing, since $F(\cdot)$ is continuous, hence

$F^{-1}(t) = G^{-1}(t)$ almost everywhere on $(0,1)$.

or, equivalently,

$$F = G.$$

The proof is complete.

3. CHARACTERIZATION THROUGH CONDITIONAL EXPECTATION

Exponential distribution

Theorem 3.1: (Lee, 2001)

If $F(x)$ is absolutely continuous with $F(x) < 1$ for all x and $c > 0$, $n \geq m+1$, then

$$E\,[X_{u(n+1)} - X_{u(n)} \mid X_{u(m)} = y] = c \tag{3.1}$$

$$E\,[X_{u(n+2)} - X_{u(n)} \mid X_{u(m)} = y] = 2c \tag{3.2}$$

$$E[X_{u(n+3)} - X_{u(n)} \mid X_{u(m)} = y] = 3c, \tag{3.3}$$

if and only if

$$F(x) = 1 - e^{-x/c}, \quad x > 0.$$

Proof: If $X_k \in \mathrm{EXP}(c)$, then

$E\,[X_{u(n)} \mid X_{u(m)} = y] = y + (n-m)c$. Hence (3.1) holds.

Conversely, suppose (3.1) holds. Using Ahsanullah formula (1995) it follows the following equation,

$$\frac{1}{1-F(y)}\int_y^\infty \frac{1}{(n-m)!}\left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m} x f(x)dx$$

$$-\frac{1}{1-F(y)}\int_y^\infty \frac{1}{(n-m-1)!}\left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m-1} x f(x)dx = c, \tag{3.4}$$

where, $c > 0$, $n \geq m+1$.

That is,

$$\frac{1}{(n-m)!}\int_y^{\infty}\left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m} x f(x)dx$$

$$-\frac{1}{(n-m-1)!}\int_y^{\infty}\left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m-1} x f(x)dx = c[1-F(y)] \tag{3.5}$$

Since $F(x)$ is absolutely continuous, we can differentiate $(n-m+1)$ times both sides of (3.5) with respect to y and simplify, then we obtain the following equation

$c\, f(y)+F(y)=1$, that is

$$f(y)+\frac{1}{c}F(y)=\frac{1}{c} \tag{3.6}$$

When we solve differential equation of (3.6), we get

$F(y)=1-e^{-y/c}$.

This completes the proof.

(3.2) and (3.3) can be proved on similar lines.

Theorem 3.2: (Lee and Lim, 2009)

Let $\{X_n, n\geq 1\}$ be a sequence of *i.i.d.* random variables with common distribution function $F(x)$ which is absolutely continuous with *pdf* $f(x)$

and $E[X_n^2]<\infty$, then

$$F(x)=1-e^{-\lambda x},\ \lambda>0,\ x>0$$

if and only if

$$E[X_{U(n+2)} \mid X_{U(n)}=y]=y+\frac{2}{\lambda}. \tag{3.7}$$

Proof: If $F(x)=1-e^{-\lambda x}$, $\lambda>0$, $x>0$, then

$$E[X_{U(n+2)} \mid X_{U(n)} = y] = e^{\lambda y} \int_y^{\infty} \left(\ln \frac{e^{\lambda x}}{e^{\lambda y}} \right) x(\lambda e^{\lambda x}) dx$$

$$= y + \frac{2}{\lambda}. \tag{3.8}$$

Hence (3.7) holds. Conversely, suppose (3.7) holds. From Ahsanullah formula (1995), we can obtain the following equation

$$\frac{1}{1-F(y)} \int_y^{\infty} \left(\ln \frac{1-F(y)}{1-F(x)} \right) x f(x) dx = y + \frac{2}{y}, \text{ for } \lambda > 0. \tag{3.9}$$

Since $F(x)$ is absolutely continuous, we can differentiate both sides of (3.9) with respect to y and simplify and we get the following equation

$$3[1-F(y)]f^2(y) - \frac{2}{\lambda} f^3(y) + [1-F(y)]^2 f'(y) \tag{3.10}$$

Let $y = F(y)$ $(i.e. y' = f(y),\ y'' = f'(y))$. Then (3.10) expressed by the following form

$$3(1-y)y'^2 - \frac{2}{\lambda}(y')^3 + (1-y)^2 y'' = 0 \tag{3.11}$$

Therefore there exists a unique solution of the differential equation (3.11) that sat satisfies the initial conditions $y(0) = 0$, $y'(0) = \lambda$ and $y''(0) = -\lambda^2$. By the existence and uniqueness theorem, we get

$$F(x) = 1 - e^{-\lambda x}.$$

This completes the proof.

Pareto distribution

Theorem 3.3: (Lee, 2003)

$F(x) = 1 - x^{\theta}, x > 0,\ \theta < -1$ if and only if

(i) $(\theta+1)E\,[X_{u(n+1)}|X_{u(m)} = y]$

$$= \theta E\,[X_{u(n)}|X_{u(m)} = y],\quad n \geq m+1 \tag{3.12}$$

(ii) $(\theta+1)^2 E\,[X_{u(n+2)}|X_{u(m)} = y]$

$$= \theta^2 E\,[X_{u(n)}|X_{u(m)} = y],\quad n \geq m+1 \tag{3.13}$$

(iii) $(\theta+1)^3 E\,[X_{u(n+3)}|X_{u(m)} = y]$

$$= \theta^3 E\,[X_{u(n)}|X_{u(m)} = y],\quad n \geq m+1 \tag{3.14}$$

Proof: If $F(x) = 1 - x^{\theta}$, then $E[X_{u(n)}|X_{u(m)} = y] = \left(\frac{\theta}{\theta+1}\right)^{n-m} y.$

Hence (3.12) holds.

Conversely, suppose (3.12) holds, then from Ahsanullah formula (1995) we can obtain the following equation,

$$\frac{\theta+1}{(n-m)!}\int_y^{\infty}\left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m} x f(x)dx$$

$$= \frac{\theta}{(n-m-1)!}\int_y^{\infty}\left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m-1} x f(x)dx \tag{3.15}$$

Since $F(x)$ is absolutely continuous, we can differentiate $(n-m+1)$ times both side of (3.15) with respect to y and simplify, then we obtain the following equation,

$$-y f(y) = \theta(1-F(y)),$$

that is, $-\dfrac{f(y)}{1-F(y)}=\dfrac{\theta}{y}$. (3.16)

Integrating both side of (3.16) with respect to y, we get,

$$F(y)=1-y^{\theta}.$$

This completes the proof.

(3.13) and (3.14) can be proved on the lines of (3.12).

General form of distributions

(a) $\bar{F}(x)=[ax+b]^{c},\ x\in(\alpha,\beta)$

Theorem 3.4: (Athar *et al.*, 2003)

Let X be an absolutely continuous *rv* with *df* $F(x)$ and *pdf* $f(x)$ on the support (α,β), where $\alpha\in\Re$ and β may be finite or infinite. Then for $r<s$,

$$E[X_{u(s)}\mid X_{u(r)}=x]=a^{*}x+b^{*} \tag{3.17}$$

if and only if

$$\bar{F}(x)=[ax+b]^{c},\ x\in(\alpha,\beta) \tag{3.18}$$

where

$$a^{*}=\left(\frac{c}{c+1}\right)^{s-r} \text{ and } b^{*}=-\frac{b}{a}[1-a^{*}].$$

Proof: First we will prove (3.18) implies (3.17).

We have

$$\bar{F}(x)=[ax+b]^{c},\ f(x)=-ac[ax+b]^{c-1}$$

Now,

$$E[X_{u(s)}\mid X_{u(r)}=x]$$

$$=\frac{1}{\Gamma(s-r)}\int_{x}^{\beta}y\left[c\ln\left(\frac{ax+b}{ay+b}\right)\right]^{s-r-1}ac[ay+b]^{c-1}dy$$

$$= \frac{1}{\Gamma(s-r)[ax+b]^c} \int_x^\beta y \left[c \ln\left(\frac{ax+b}{ay+b}\right)\right]^{s-r-1} ac[ay+b]^{c-1} dy$$

Let $t = \ln\left(\frac{ax+b}{ay+b}\right)^c$, then R.H.S. is

$$\frac{1}{a\Gamma(s-r)} \int_0^\infty [(ax+b)e^{-t/c} - b] t^{s-r-1} e^{-t} dt$$

which reduces to,

$$\left(\frac{c}{c+1}\right)^{s-r} x + \frac{b}{a}\left[\left(\frac{c}{c+1}\right)^{s-r} - 1\right] \tag{3.19}$$

and hence the 'if' part.

To prove (3.17) implies (3.18), we have

$$\frac{1}{\Gamma(s-r)} \int_x^\beta y[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-r-1} \frac{f(y)}{\overline{F}(x)} dy = a^* x + b^*$$

Now set $\overline{F}(x) = e^{-v}$ and $\overline{F}(y) = e^{-(u+v)}$, then

$$\frac{1}{\Gamma(s-r)} \int_0^\infty \overline{F}^{-1}[e^{-(u+v)}] u^{s-r-1} e^{-u} du = a^* \overline{F}^{-1}(e^{-v}) + b^*$$

Set $G(v) = \overline{F}^{-1}(e^{-v})$, to get

$$\int G(u+v)\, \mu(du) = G(v) + \frac{b^*}{a^*} \tag{3.20}$$

where

$$\mu(du) = \frac{1}{\Gamma(s-r)a^*} u^{s-r-1} e^{-u} du \tag{3.21}$$

Now, since (3.20) is of the form (2.26), and therefore in view of (2.26), (2.27) and (2.28), we get

$$G(v) = \gamma + \alpha'(1 - e^{\eta v}), \text{ if } \eta \neq 0$$

Thus at $\eta \neq 0$,

$$G(v) = \bar{F}^{-1}(e^{-v}) = \gamma + \alpha'(1 - e^{\eta v})$$

or,

$$e^{-v} = \bar{F}[\gamma + \alpha'(1 - e^{\eta v})]$$

Let $z = \gamma + \alpha'(1 - e^{\eta v})$ and therefore,

$$\bar{F}(z) = \left[1 - \frac{z - \gamma}{\alpha'}\right]^{-\frac{1}{\eta}} = [az + b]^{c}$$

where

$$a = -\frac{1}{\alpha'},\ b = \frac{\alpha' + \gamma}{\alpha'},\ c = -\frac{1}{\eta} \tag{3.22}$$

In view of (2.28) and alongwith (3.21) and (3.22), we get

$$a^{*} = \frac{1}{(1-\eta)^{s-r}} = \left(\frac{c}{c+1}\right)^{s-r} \tag{3.23}$$

Further at $v = 0$, we have from (2.26) and (2.27),

$$\int G(u)\mu(du) = G(0) + c^{*}$$

or,

$$G(0) + c^{*} = \int(\gamma + \alpha' - \alpha' e^{\eta x})\mu(dx)$$

i.e. $$\gamma + \frac{b^{*}}{a^{*}} = \frac{\gamma + \alpha'}{a^{*}} - \alpha' \Leftrightarrow (\gamma + \alpha') = \frac{b^{*}}{1 - a^{*}}$$

Using (3.22)

$$(\gamma + \alpha') = -\frac{b}{a} = \frac{b^{*}}{1 - a^{*}}$$

and hence

$$b^{*} = -\frac{b}{a}[1 - a^{*}]. \tag{3.24}$$

For $\eta = 0$, from (2.27),

$$G(v) = \overline{F}^{-1}(e^{-v}) = \gamma + \beta' v$$

or,

$$\overline{F}(z) = e^{-\lambda(z-\gamma)}, \ \lambda > 0 \tag{3.25}$$

where $\lambda = \dfrac{1}{\beta'}$. Therefore

$$\overline{F}(z) = \left[1 - \frac{\lambda(z-\gamma)}{c}\right]^c = [az+b]^c \text{ as } c \to \infty (\eta \to 0)$$

where

$$a = -\frac{\lambda}{c}, \ b = \frac{c + \lambda\gamma}{c}. \text{ So}$$

$$a^* = \left(\frac{c}{c+1}\right)^{s-r} \to 1 \text{ as } c \to \infty \tag{3.26}$$

and

$$b^* = \frac{b}{a}\left[\left(\frac{c}{c+1}\right)^{s-r} - 1\right]$$

Therefore,

$$\frac{b}{a} = -\frac{c + \lambda\gamma}{\lambda} = -\frac{t(1-\lambda\gamma) + \lambda\gamma}{(1-t)\lambda}$$

and with $t = \dfrac{c}{c+1}$

$$b^* = \frac{t^{s-r+1}(1-\lambda\gamma) + t^{s-r}\lambda\gamma - t(1-\lambda\gamma) - \lambda\gamma}{(t-1)\lambda}$$

Differentiating numerator and denominator separately *w.r.t.* t and taking limit as $t \to 1$, we get

$$b^* = \frac{(s-r)}{\lambda} \tag{3.27}$$

The values of a^* and b^* at $\eta = 0$ could have also been obtained from (2.27), (2.28), (3.20) and (3.21) as earlier and hence the Theorem.

(i) Power function distribution

$$\bar{F}(x) = \left(\frac{\beta - x}{\beta - \alpha}\right)^{\theta} = \left[-\frac{1}{\beta-\alpha}x + \frac{\beta}{\beta-\alpha}\right]^{\theta}, \ \alpha \le x \le \beta \tag{3.28}$$

$$a = -\frac{1}{\beta-\alpha}, \ b = \frac{\beta}{\beta-\alpha}, \ c = \theta,$$

$$a^* = \left(\frac{\theta}{\theta-1}\right)^{s-r} < 1, \ b^* = \beta\left[1 - \left(\frac{\theta}{\theta-1}\right)^{s-r}\right] \tag{3.29}$$

(ii) Pareto distribution

$$\bar{F}(x) = \left(\frac{\alpha+\delta}{x+\delta}\right)^{\theta} = \left[\frac{1}{\alpha+\delta}x + \frac{\delta}{\alpha+\delta}\right]^{-\theta},$$

$$\alpha \le x < \infty, \theta > 0, \alpha + \delta > 0 \tag{3.30}$$

$$a = -\frac{1}{\alpha+\delta}, \ b = \frac{\delta}{\alpha+\delta}, \ c = -\theta,$$

$$a^* = \left(\frac{\theta}{\theta-1}\right)^{s-r} < 1, \ b^* = \delta\left[\left(\frac{\theta}{\theta-1}\right)^{s-r} - 1\right] \tag{3.31}$$

(iii) Exponential distribution

$$\bar{F}(x) = e^{-\lambda(x-\alpha)}, \ x \ge \alpha, \ \lambda > 0$$

$$a = -\frac{\lambda}{c}, \ b = \frac{c + \lambda\alpha}{c}, \ c \to \infty \tag{3.32}$$

$$a^* = 1, \ b^* = \frac{(s-r)}{\lambda} \tag{3.33}$$

Remark 3.1: If $h(x)$ be monotonic function of x on support (α,β), then we have

$$E[h(X_{u(s)}) \mid X_{u(r)} = x] = a^* h(x) + b^*$$

if and only if

$$\bar{F}(x) = [ah(x) + b]^c$$

with the same a^* and b^* as given in (3.23) and (3.24). Therefore, with suitable choice of a, b, c and $h(x)$, we will get various characterization results of distribution [Khan and Aboummoh, 2000].

(b) $\quad F(x) = 1 - \exp\left\{-\frac{1}{c}[h(x) - h(\alpha)]\right\}, \; x \in (\alpha, \beta)$

Theorem 3.5: (Wu, 2004)

Let $\alpha(\geq -\infty)$ and $\beta(\leq \infty)$ be the left and right extremities respectively of an absolutely continuous *df* F, and let h be a monotonic function continuous on $[\alpha,\beta)$ and with (its derivative) h' defined and continuous on (α,β). We assume : $0 < F(x) < 1$, for $x \in (\alpha,\beta)$. If $E[h(X_1)] < \infty$, then

$$E[h(X_{U(j)}) \mid X_{U(i)} = x] = h(x) + (j-i)c, \; j > i \geq 1, \; x \in (\alpha,\beta) \qquad (3.34)$$

holds, if and only if

$$F(x) = 1 - \exp\left\{-\frac{1}{c}[h(x) - h(\alpha)]\right\}, \; x \in (\alpha,\beta) \qquad (3.35)$$

holds, with $h(x) \to \pm\infty$ as $x \to \beta^-$ respectively according as $c > 0$ or $c < 0$.

Proof: It can be shown (Ahsanullah, 1995) that the conditional *pdf* of $X_{U(j)}$ given $X_{U(i)} = x (j > i)$ is given by

$$f_{j,i}(y \mid x) = \frac{1}{(j-i-1)!}\{-\ln[1-F(y)] + \ln[1-F(x)]\}^{j-i-1} \frac{f(y)}{1-F(x)},$$

for $y > x$.

Therefore, for $j > i$,

$$E[h(X_{U(j)}) \mid X_{U(i)} = x]$$

$$= \int_x^\beta \frac{h(y)}{(j-i-1)!}\{-\ln[1-F(y)]+\ln[1-F(x)]\}^{j-i-1}\frac{f(y)}{1-F(x)}dy \qquad (3.36)$$

To prove the sufficient part, if $F(x)$ is given as (3.35), then

$$-\ln[1-F(x)] = \frac{1}{c}[h(x)-h(\alpha)], \quad x \in (\alpha, \beta)$$

and (3.36) can be reduced to

$$E[h(X_{U(j)}) \mid X_{U(i)} = x]$$

$$= \int_x^\beta \frac{h(y)}{(j-i-1)!}\left\{\frac{1}{c}[h(y)-h(x)]\right\}^{j-i-1} \exp\left\{-\frac{1}{c}[h(y)-h(\alpha)]\right\}\frac{1}{c}h'(y)dy$$

$$= \frac{1}{(j-i-1)![1-F(x)]}\left[\int_x^\beta h'(y)\left\{\frac{1}{c}[h(y)-h(x)]\right\}^{j-i-1}\right.$$

$$\times \exp\left\{-\frac{1}{c}[h(y)-h(\alpha)]\right\}dy + (j-i-1)\int_x^\beta h(y)\left\{\frac{1}{c}[h(y)-h(x)]\right\}^{j-i-2}$$

$$\left.\times \exp\left\{-\frac{1}{c}[h(y)-h(\alpha)]\right\}\frac{1}{c}h'(y)dy\right]$$

(by integration by parts and $E[h(X_1)] < \infty$)

$$= \frac{1}{(j-i-1)![1-F(x)]}\left[\int_x^\beta h'(y)\left\{\frac{1}{c}[h(y)-h(x)]\right\}^{j-i-1}\right.$$

$$\left.\times \exp\left\{-\frac{1}{c}[h(y)-h(\alpha)]\right\}dy\right] + E[h(X_{U(j-1)}) \mid X_{U(i)} = x]$$

$$= \frac{1}{[1-F(x)]}\int_x^\beta \exp\left\{-\frac{1}{c}[h(y)-h(\alpha)]\right\}h'(y)dy$$

$$+ E[h(X_{U(j-1)}) \mid X_{U(i)} = x]$$

(on $(j-i-1)$ times of integration by parts and $E[h(X_1)] < \infty$)

$$= c + E[h(X_{U(j-1)}) \mid X_{U(i)} = x].$$

Hence, by using the above recurrence equation successively, we obtain

$$E[h(X_{U(j)}) \mid X_{U(i)} = x] = (j-i)c + h(x), \;\; j > i.$$

Next, we prove the necessary part. If (3.35) holds, then from (3.37), we have

$$(j-i)c + h(x) = \frac{1}{(j-i-1)!} \int_x^\beta h(y)\{-\ln[1-F(y)] + \ln[1-F(x)]\}^{j-i-1} \times \frac{f(y)}{1-F(x)} dy$$

or, equivalently,

$$[(j-i)c + h(x)][1-F(x)] = \frac{1}{(j-i-1)!} \int_x^\beta h(y)\{-\ln[1-F(y)] + \ln[1-F(x)]\}^{j-i-1} \times f(y) dy \tag{3.37}$$

By differentiating (3.37) *w.r.t* x, we obtain

$$\frac{-1}{(j-i-1)!} \int_x^\beta (j-i-1) h(y)\{-\ln[1-F(y)] + \ln[1-F(x)]\}^{j-i-2} \frac{f(x)}{1-F(x)} dy$$

$$= -[(j-i-1)c + h(x)] f(x) + h'(x)[1-F(x)] \tag{3.38}$$

If we replace j by $(j-1)$ in (3.37), then (3.38) can be reduced to

$$-f(x)[(j-i-1)c + h(x)] = -f(x)[(j-i)c + h(x)] + h'(x)[1-F(x)] \tag{3.39}$$

In (3.39), we get after some calculations

$$cf(x) = h'(x)[1-F(x)],$$

or, equivalently,

$$\frac{d[1-F(x)]}{1-F(x)} = -\frac{h'(x)}{c} dx \tag{3.40}$$

Integrating (3.40) *w.r.t.* x from α to x, $x \in (\alpha, \beta)$, the assumption of absolute continuity of F and $F(\alpha) = 0$, yields

$$F(x) = 1 - \exp\left\{-\frac{1}{c}[h(x) - h(\alpha)]\right\}, \ x \in (\alpha, \beta),$$

which is the desired result.

Remark 3.2: if we take $i = j-1, \ j > 1,$ then Theorem 3.5 reduces to the result of Nagraja (1977).

Remark 3.3: if we take $h(x) = x$ and $j = i+1, \ i \geq 1,$ then Theorem 3.5 reduces to the result of Nagraja (1988b).

Remark 3.4: if we take $h(x) = x$ and $j = m+2, \ i = m,$ then Theorem 3.5 reduces to the result 3 of Theorem 2.1 of Ahsanullah and Wesolowski (1998).

(c) $\quad \overline{F}(x) = e^{-h(x)/c}, \ c > 0$

Theorem 3.6: (Khan *et al.*, 2010a)

Let X be an absolutely continuous random variable with the *df* $F(x)$ and *pdf* $f(x)$ on the support (α, β), where α and β may be finite or infinite. Then for $m \leq r < s$

$$E\,[h(X_{u(s)}) - h(X_{u(r)}) \mid h(X_{u(m)}) = x] = (s-r)c \tag{3.41}$$

if and only if

$$\overline{F}(x) = e^{-h(x)/c}, \ c > 0 \tag{3.42}$$

where $h(x)$ is a monotonic and differentiable function of x such that $h(x) \to 0$ as $x \to \alpha$ and $h(x)\overline{F}(x) \to 0$ as $x \to \beta$.

Proof: we have,

$$E\,[h(X_{u(s)}) - h(X_{u(r)}) \mid h(X_{u(m)}) = x]$$

$$= \frac{1}{\Gamma(s-m)} \int_x^\beta h(y)[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-m-1} \frac{f(y)}{\overline{F}(x)} dy$$

$$- \frac{1}{\Gamma(r-m)} \int_x^\beta h(y)[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{r-m-1} \frac{f(y)}{\overline{F}(x)} dy \tag{3.43}$$

Now, it is easy to see that (3.42) implies (3.41).

For sufficiency part, let $c^* = (s-r)c$, then

$$\frac{1}{\Gamma(s-m)}\int_x^\beta h(y)[-\ln\overline{F}(y)+\ln\overline{F}(x)]^{s-m-1}f(y)dy$$

$$-\frac{1}{\Gamma(r-m)}\int_x^\beta h(y)[-\ln\overline{F}(y)+\ln\overline{F}(x)]^{r-m-1}f(y)dy = c^*\,\overline{F}(x) \qquad (3.44)$$

Differentiating $(r-m)$ times both the sides of (3.44) with respect to x, we get

$$\frac{1}{\Gamma(s-r)}\int_x^\beta h(y)[-\ln\overline{F}(y)+\ln\overline{F}(x)]^{s-r-1}\frac{f(y)}{\overline{F}(x)}dy = h(x)+c^* \qquad (3.45)$$

Integrating LHS of (3.45) by parts and simplifying, we have

$$\frac{1}{\Gamma(s-r-1)[\overline{F}(x)]}\int_x^\beta h(y)[-\ln\overline{F}(y)+\ln\overline{F}(x)]^{s-r-2}f(y)dy$$

$$+\frac{1}{\Gamma(s-r)[\overline{F}(x)]}\int_x^\beta h'(y)[-\ln\overline{F}(y)+\ln\overline{F}(x)]^{s-r-1}\overline{F}(y)dy$$

$$= h(x)+c^* \qquad (3.46)$$

This in view of (3.45), reduces to

$$\frac{1}{\Gamma(s-r)}\int_x^\beta h'(y)[-\ln\overline{F}(y)+\ln\overline{F}(x)]^{s-r-1}\overline{F}(y)dy = c\,\overline{F}(x) \qquad (3.47)$$

Differentiating (3.48) $(s-r)$ times with respect to x, we obtain

$$h'(x)\overline{F}(x) = c\,f(x)$$

and hence the result.

Remark 3.5: At $s=r+1$, $s=r+2$ and $h(x)=x$, we get the result as obtained by Lee (2001).

Remark 3.6: At $s=r+3$, $s=r+4$ and $h(x)=x$, this reduces to the result as obtained by Lee *at al.*(2002).

Remark 3.7: At $r=m$, $E[h(X_{u(s)}\mid X_{u(r)}=x]=h(x)+(s-r)c$ as obtained by Athar *et al.* (2003).

4. CHARACTERIZATION THROUGH DISTRIBUTIONAL PROPERTY

Exponential distribution

Theorem 4.1: (Ahsanullah, 1991)

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed non-negative random variables with absolutely continuous (with respect to Lebesgue measure) distribution function $F(x)$ and the corresponding probability density function $f(x)$. If $F \in c_2$ and for some $m, m \geq 2$,

$$X_{L(m)} \stackrel{d}{=} X_{L(m-1)} + U,$$

where U is independent of $X_{L(m)}$ and $X_{L(m-1)}$ and is identically distributed as X_i's, then $X \in E(\sigma)$, for some $\sigma > 0$.

Proof: Suppose

$$R(y) = \int_0^y r(x)dx, \quad 0 < y < \infty,$$

then for $0 < y < \infty$ the *pdf* $f_1(y)$ of $X_{L(m)}$ $(m \geq 2)$ can be written as

$$f_1(y) = \frac{[R(y)]^{m-1}}{(m-1)!} f(y);$$

$$= \frac{d}{dy}\left(-\bar{F}(y) \int_0^y \frac{[R(x)]^{m-2}}{(m-2)!} r(x)dx + \int_0^y \frac{[R(x)]^{m-1}}{(m-2)!} f(x)dx \right) \quad (4.1)$$

The *pdf* $f_2(y)$ of $X_{L(m-1)} + U$ can be written as

$$f_2(y) = \int_0^y \frac{[R(y-x)]^{m-2}}{(m-2)!} f(y-x)f(x)dx$$

$$= \frac{d}{dy}\left(-\int_0^y \frac{[R(x)]^{m-2}}{(m-2)!} \bar{F}(y-x)f(x)dx + \int_0^y \frac{[R(x)]^{m-2}}{(m-2)!} f(x)dx \right) \quad (4.2)$$

From equation (4.1) and (4.2), on simplification we get

$$\int_0^y \frac{[R(x)]^{m-2}}{(m-2)!} f(x)H_1(x,y)dx = 0, \quad (4.3)$$

where,

$$H_1(x,y) = \overline{F}(y-x) - \overline{F}(y)(\overline{F}(x))^{-1},\ 0 < x < y < \infty.$$

Since $F \in c_2$, for (4.3) to be true, we must have

$$H_1(x,y) = 0 \text{ for almost all } x, 0 < x < y. \tag{4.4}$$

Now $H_1(x,y) = 0$ for almost all $x, 0 < x < y$ implies

$$\overline{F}(y-x)\overline{F}(x) = \overline{F}(y), \text{ for almost all } x, 0 < x < y < \infty \tag{4.5}$$

The only continuous solution of (4.5) with the boundary conditions $\overline{F}(0) = 1$ and $\overline{F}(\infty) = 0$, is

$$\overline{F}(x) = e^{-\theta x}, \text{ where } \theta \text{ is an arbitrary positive number.} \tag{4.6}$$

Theorem 4.2: (Ahsanullah, 1991)

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed non-negative random variables with absolutely continuous (with respect to Lebesgue measure) distribution function $F(x)$ and the corresponding probability density function $f(x)$. If $F \in c_1$ and for some m, n with $1 \leq m < n$,

$$E[X_{L(n)} - X_{L(m)}] \underline{\underline{d}}\ E[X_{L(n-m)}],$$

then $X_n \in E(\sigma)$, for some $\sigma > 0$.

Proof : See Reference.

Theorem 4.3: (Ahsanullah, 2010)

Let $X_1, X_2, \ldots$ be a sequence of non-negative *i.i.d* random variables with *cdf* $F(x)$ and *pdf* $f(x)$. We assume $F(0) = 0$ and $F(x) < 1$ for all $x > 0$ and $E[X^k]$ exists for some $k > 0$. Then the following statements are equivalent

(a) F is $E(\theta)$

(b) $X_{U(n)} \underline{\underline{d}}\ X_{U(n-1)} + W$,, where $F \in C$, W is independent of $X_{U(n-1)}$ and is distributed as F for any fixed $n > 1$.

Proof: See reference.

Pareto distribution

Theorem 4.4: (Chang, 2007)

Let $\{X_n, n \geq 1\}$ be a sequence of *i.i.d.* random variables with *cdf* $F(x)$ which is an absolutely continuous with *pdf* $f(x)$ and $F(1)=0$ and $F(x)<1$ for all $x>1$. Then $F(x)=1-x^{-\beta}$ for all $x>1$ and $\beta>0$, if and only if $\dfrac{X_{U(n)}}{X_{U(n)}-X_{U(n+1)}}$ and $X_{U(n)}$ are independent for $n \geq 1$.

Proof: If $F(x)=1-x^{-\beta}$ for all $x>1$ and $\beta>0$, then the joint *pdf* $f_{n,n+1}(x,y)$ of $X_{U(n)}$ and $X_{U(n+1)}$ is

$$f_{n,n+1}(x,y)=\frac{\beta^{n+1}(\ln x)^{n-1}}{\Gamma(n)x\,y^{\beta+1}}$$

for all $1<x<y$, $\beta>0$ and $n \geq 1$.

Consider the function $V=\dfrac{X_{U(n)}}{X_{U(n)}-X_{U(n+1)}}$ and $W=X_{U(n)}$. It follows that $x_{U(n)}=w$, $x_{U(n+1)}=\dfrac{(v-1)w}{v}$ and $|J|=\dfrac{w}{v^2}$. Thus we can write the joint *pdf* $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w)=\frac{\beta^{n+1}\,v^{\beta-1}(\ln w)^{n-1}}{\Gamma(n)(v-1)^{\beta+1}\,w^{\beta+1}} \tag{4.7}$$

for all $v>0$, $w>1$, $\beta>0$ and $n \geq 1$.

The marginal *pdf* $f_V(v)$ of V is given by

$$f_V(v)=\int_1^\infty f_{V,W}(v,w)\,dw$$

$$=\frac{\beta^{n+1}\,v^{\beta-1}}{\Gamma(n)(v-1)^{\beta+1}}\int_1^\infty \frac{(\ln w)^{n-1}}{w^{\beta+1}}dw$$

$$= \frac{\beta v^{\beta-1}}{(v-1)^{\beta+1}} \tag{4.8}$$

for all $v > 0$, $\beta > 0$ and $n \geq 1$.

Also, the *pdf* $f_W(w)$ of W is given by

$$f_W(w) = \frac{(R(w))^{n-1} f(w)}{\Gamma(n)}$$

$$= \frac{\beta^n (\ln w)^{n-1}}{\Gamma(n) w^{\beta+1}} \tag{4.9}$$

for all $w > 1$, $\beta > 0$ and $n \geq 1$.

From (4.7), (4.8) and (4.9), we obtain $f_V(v)\, f_W(w) = f_{V,W}(v,w)$.

Hence $V = \dfrac{X_{U(n)}}{X_{U(n)} - X_{U(n+1)}}$ and $W = X_{U(n)}$ are independent for $n \geq 1$.

Now we will prove the sufficient condition. The joint *pdf* $f_{n,n+1}(x,y)$ of $X_{U(n)}$ and $X_{U(n+1)}$ is

$$f_{n,n+1}(x,y) = \frac{[R(x)]^{n-1} r(x) f(y)}{\Gamma(n)}$$

for all $1 < x < y$, $\beta > 0$ and $n \geq 1$, where

$$R(x) = -\ln(1 - F(x)) \text{ and } r(x) = \frac{d}{dx}(R(x)) = \frac{f(x)}{1 - F(x)}$$

Let us use the transformation $V = \dfrac{X_{U(n)}}{X_{U(n)} - X_{U(n+1)}}$ and $W = X_{U(n)}$.

The jacobian of this transformation is $|J| = \dfrac{w}{v^2}$. Thus we can write the *pdf* of $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w) = \frac{f\left(\frac{(v-1)w}{v}\right)(R(w))^{n-1} r(w) w}{\Gamma(n) v^2} \tag{4.10}$$

for all $v > 0$, $w > 1$ and $n \geq 1$.

The *pdf* $f_W(w)$ of W is given by

$$f_W(w) = \frac{[R(w)]^{n-1} f(w)}{\Gamma(n)} \tag{4.11}$$

for all $w > 0$ and $n \geq 1$.

From (4.10) and (4.11), we obtain the *pdf* $f_V(v)$ of V

$$f_V(v) = \frac{f\left(\frac{(v-1)w}{v}\right) r(w) w}{v^2 f(w)}$$

for all $v > 0$, $w > 1$ and $n \geq 1$, where $r(x) = \frac{f(x)}{1-F(x)}$.

That is,

$$f_V(v) = \frac{\partial}{\partial v}\left(\frac{1-F\left(\frac{(v-1)w}{v}\right)}{1-F(w)}\right).$$

Since V and W are independent, we must have

$$1-F\left(\frac{(v-1)w}{v}\right) = \left(1-F\left(\frac{(v-1)}{v}\right)\right)(1-F(w)) \tag{4.12}$$

for all $v > 0$ and $w > 1$.

Upon substituting $\frac{(v-1)}{v} = v_1$ in (4.12), then we get

$$1-F(v_1 w) = (1-F(v_1))(1-F(w)) \tag{4.13}$$

for all $v_1 > 1$ and $w > 1$.

By the monotonic transformation of the exponential distribution, the most general solution of (4.13) with the boundary condition $F(1) = 0$ is

$$F(x) = 1 - x^{-\beta}$$

for all $x > 1$ and $\beta > 0$.

This completes the proof.

Theorem 4.5: (Chang, 2007)

Let $\{X_n, n \geq 1\}$ be a sequence of *i.i.d.* random variables with *cdf* $F(x)$ which is an absolutely continuous with *pdf* $f(x)$ and $F(1) = 0$ and $F(x) < 1$ for all $x > 1$. Then $F(x) = 1 - x^{-\beta}$ for all $x > 1$ and $\beta > 0$, if and only if $\dfrac{X_{U(n)} - X_{U(n+1)}}{X_{U(n)}}$ and $X_{U(n)}$ are independent for $n \geq 1$.

Proof: See reference.

Gumbel distribution

Theorem 4.6: (Alzaid and Ahsanullah, 2003)

Let $\{X_j, j = 1, 2, ...\}$ be a sequence of independent and identically distributed random variables with absolutely continuous (with respect to Lebesgue measure) distribution function $F(x)$. Then the following two statements are identical.

(a) $F(x) = e^{-e^{-x}}, -\infty < x < \infty,$

(b) For a fixed $m > 1$, the condition $-\ln F(0) = 1$ and $X_{L(m)} \underset{=}{d} X_{L(m+1)} + W/m,$

where W is distributed as exponential with unit mean.

Proof: It is enough to show that (b)$\Rightarrow (a)$.

Suppose that for a fixed $m > 1$, $X_{L(m)} \underset{=}{d} X_{L(m+1)} + W/m$, then

$$F_{(m)}(x) = \int_{-\infty}^{x} P[W \le m(x-y)] f_{(m+1)}(y)dy$$

$$= \int_{-\infty}^{x} [1 - e^{-m(x-y)}] f_{(m+1)}(y)dy$$

$$= F_{(m+1)}(x) - \int_{-\infty}^{x} e^{-m(x-y)} f_{(m+1)}(y)dy \tag{4.14}$$

Thus,

$$e^{mx}[F_{(m+1)}(x) - F_{(m)}(x)] = \int_{-\infty}^{x} e^{mx} f_{(m+1)}(y)dy \tag{4.15}$$

Using the relation following relation,

$$\overline{F}_{(n)}(x) = 1 - F_{(n)}(x) = e^{-H(x)} \sum_{j=0}^{n-1} \frac{[H(x)]^j}{j!}$$

we obtain,

$$e^{m(x)} \frac{F(x)[H(x)]^m}{\Gamma(m+1)} = \int_{-\infty}^{x} e^{my} f_{(m+1)}(x) \tag{4.16}$$

Taking the derivatives of both sides of equation (4.16), we obtain

$$\frac{d}{dx}\left[e^{mx} \frac{[H(x)]^m}{\Gamma(m+1)} F(x) \right] = e^{mx} f_{(m+1)}(x) \tag{4.17}$$

This implies that

$$\frac{d}{dx}\left[e^{mx} \frac{[H(x)]^m}{\Gamma(m+1)} \right] F(x) = 0 \tag{4.18}$$

Thus

$$\frac{d}{dx}\left[e^{mx} \frac{[H(x)]^m}{\Gamma(m+1)} \right] = 0 \tag{4.19}$$

Hence

$$H(x) = ce^{-x}, \; -\infty < x < \infty \tag{4.20}$$

Thus, we obtain

$$F(x) = e^{-ce^{-x}}, -\infty < x < \infty. \tag{4.21}$$

Since $F(x)$ is a distribution function we must have c as positive. Using the condition $-\ln F(0) = 1$, we obtain

$$F(x) = e^{-e^{-x}}, -\infty < x < \infty. \tag{4.22}$$

Theorem 4.7: (Alzaid and Ahsanullah, 2003)

Let X be a non-negative random variables having an absolutely continuous (with respect to Lebesgue measure) distribution function $F(x)$. Then the following two properties are equivalent.

(a) X has an exponential distribution with $F(x) = 1 - e^{-x}$, $x > 0$.

(b) For some fixed $r > 1$, $X_{r:n} \underline{\underline{d}} X_{r-1:n} + W/(n-r+1)$ where W is distributed as exponential.

Proof: See reference.

CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS THROUGH GENERALIZED ORDER STATISTICS

1. INTRODUCTION

Characterization through conditional moments of generalized order statistics is of special interest. Keseling (1999) has generalized the result of Franco and Ruiz (1995) in terms of generalized order statistics (*gos*) and characterized some general class of distributions.

Kamps (1996) characterized uniform distribution by sub-ranges and extended to generalized order statistics. Mahmoud and Al-Nagar (2007) characterized the generalized power function distribution based on the conditional moments of the generalized order statistics.

Khan and Alzaid (2004) characterized a general class of distributions $\overline{F}(x) = (ax+b)^c$ through linear regression of generalized order statistics using Rao and Shanbhag's (1994) result. They characterized distributions by using the relation

$$E\,[X(s,n,m,k) \mid X(r,n,m,k) = x] = a^* h(x) + b^* .$$

Further, Khan *et al.* (2006) have characterized distributions by means of the relation

$$E[\xi\{X(s,n,m,k)\} \mid X(r,n,m,k) = x] = g_{s|r}(x)$$

and its dual

$$E[\xi\{X(r,n,m,k)\} \mid X(s,n,m,k) = x] = g_{r|s}(x) .$$

Samuel (2008) characterized general forms of several well known continuous probability distributions by the conditional expectation of certain functions of generalized order statistics.

Haque and Faizan (2010) characterized Weibull distribution through conditional variance of generalized order statistics (*gos*) whereas Athar *et al.* (2010) characterized a general class of distribution

$$\overline{F}(x) = \exp[-\frac{1}{c}\{h(x) - h(\alpha)\}],\ x \in (\alpha, \beta)$$

through generalized order statistics (*gos*).

Characterization of probability distributions through distributional properties of generalized order statistics have been considered by many authors in literature. Kamps and Gather (1997) characterized the exponential distribution whereas Ahsanullah (2000) characterized the exponential distribution based on independence of functions of generalized order statistics and presented the estimators of its parameters. Tawangar and Asadi (2007) characterized the generalized pareto distribution (*GPD*) and uniform distribution based on generalized order statistics and the results are linked with the result of Oakes and Dasu (1990) and extend some of the results obtained by Asadi and Bayramogulu (2006).

For more results on characterization through generalized order statistics one may refer to Ahsanullah (1997), Cramer and Kamps (2000), Ahmad and Fawzy (2003), Ahsanullah (2004) and references therein.

2. CHARACTERIZATION THROUGH MOMENTS

Generalized Power function distribution

The standard generalized power function distribution has probability density function (*pdf*) $f(x)$ and distribution function $F(x)$ as follows

$$f(x) = \frac{p}{b^p}(x+a)^{p-1}, \quad -a \le x \le b-a \tag{2.1}$$

and

$$F(x) = \frac{1}{b^p}(x+a)^p, \tag{2.2}$$

where, $p \geq 1$, $a = \sqrt{p(p+2)}$ and $b = (p+1)\sqrt{\frac{p+2}{p}}$.

Then,

$$f(x) = \frac{p}{(x+a)} F(x). \tag{2.3}$$

Theorem 2.1: (Mahmoud and Al-Nagar, 2007)

Let $X(1,n,\tilde{m},k)$ and $X(2,n,\tilde{m},k)$ be generalized order statistics based on continuous distribution function $F(x)$. Then the conditional expectation of generalized order statistics $X(2,n,\tilde{m},k)$ given $X(1,n,\tilde{m},k)$ is given by

$$E[X(2,n,\tilde{m},k) \mid X(1,n,\tilde{m},k) = x] = x[1-F(x)]^{m_1-\gamma_1+\gamma_2+1}$$

$$+[1-F(x)]^{m_1-\gamma_1+1} \sum_{j=0}^{\gamma_2} (-1)^j \binom{\gamma_2}{j} \frac{b^{p_j+1} - (x+a)^{p_j+1}}{b^{p_j}(p_j+1)} \tag{2.4}$$

if and only if X has a generalized power function distribution (2.1).

Proof: First , we prove that if X has a generalized power function distribution (2.1) then (2.4) holds.

Setting $r = 1$ and $s = 2$ in (4.4) and making use of the following equation

$$\mu^{(\alpha)}(r,n,\tilde{m},k) = \frac{c_{r-1}}{(r-1)!} \int x^j [1-F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx,$$

where

$$E[X^{\alpha}(r,n,\tilde{m},k)] = \mu^{(\alpha)}(r,n,\tilde{m},k), \text{ yields}$$

$$E[X(2,n,\tilde{m},k) \mid X(1,n,\tilde{m},k) = x] = \gamma_2 [1-F(x)]^{m_1-\gamma_1+1} \int_x^{b-a} x_2 f(x_2)$$

$$\times [1-F(x_2)]^{\gamma_2 - 1} dx_2$$

Integrating by parts gives

$$E[X(2,n,\tilde{m},k) \mid X(1,n,\tilde{m},k) = x] = x[1-F(x)]^{m_1-\gamma_1+\gamma_2+1}$$

$$+[1-F(x)]^{m_1-\gamma_1+1} \int_x^{b-a} [1-F(x_2)]^{\gamma_2} dx_2.$$

Upon using (2.2) and expanding binomially, we get

$$E[X(2,n,\tilde{m},k) \mid X(1,n,\tilde{m},k)=x] = [1-F(x)]^{m_1-\gamma_1+1}$$

$$\times \{x[1-F(x)]^{\gamma_2} + \sum_{j=0}^{\gamma_2} (-1)^j \binom{\gamma_2}{j} \frac{1}{b^{p_j}} \int_x^{b-a} (x_2+a)^{p_j}\, dx_2\}.$$

and hence the (2.4).

Now, let (2.4) hold, then

$$\gamma_2[1-F(x)]^{m_1-\gamma_1+1} \int_x^{b-a} x_2 f(x_2)[1-F(x_2)]^{\gamma_2-1} dx_2$$

$$= x[1-F(x)]^{m_1-\gamma_1+\gamma_2+1} + [1-F(x)]^{m_1-\gamma_1+1}$$

$$\times \sum_{j=0}^{\gamma_2} (-1)^j \binom{\gamma_2}{j} \frac{b^{p_j+1}-(x+a)^{p_j+1}}{b^{p_j}(p_j+1)}.$$

Differentiating both sides *w.r.t.* x and simplifying, we get

$$[1-F(x)]^{\gamma_2} = \sum_{j=0}^{\gamma_2} (-1)^j \binom{\gamma_2}{j} \frac{(x+a)^{p_j}}{b^{p_j}}.$$

Therefore

$$F(x) = \frac{1}{b^p}(x+a)^p$$

This completes the proof.

Theorem 2.2: (Mahmoud and Al-Nagar, 2007)

Let $X(1,n,\tilde{m},k)$ and $X(2,n,\tilde{m},k)$ be generalized order statistics based on continuous distribution function $F(x)$. Then the conditional expectation of generalized order statistics $X^2(2,n,\tilde{m},k)$ given $X(1,n,\tilde{m},k)$ is given by

$$E[X^2(2,n,\tilde{m},k) \mid X(1,n,\tilde{m},k)=x]$$

$$= x^2[1-F(x)]^{m_1-\gamma_1+\gamma_2+1} + 2[1-F(x)]^{m_1-\gamma_1-1}\sum_{j=0}^{\gamma_2}(-1)^j\binom{\gamma_2}{j}$$

$$\times\left\{\frac{b(b-a)}{(p_j+1)} - \frac{b^2}{(p_j+1)(p_j+2)} - \frac{x(x+a)^{p_j-1}}{b^{p_j}(p_j+1)} + \frac{(x+a)^{p_j+2}}{b^{p_j}(p_j+1)(p_j+2)}\right\} \tag{2.5}$$

if and only if X has a generalized power function distribution (2.1).

Proof: To prove necessary part, we have

we have

$$E[X^2(2,n,\tilde{m},k) \mid X(1,n,\tilde{m},k)=x] = \gamma_2[1-F(x)]^{m_1-\gamma_1+1}\int_x^{b-a} x_2^2 f(x_2)$$
$$\times[1-F(x_2)]^{\gamma_2-1}dx_2$$

Integrating by parts, we obtain

$$E[X^2(2,n,\tilde{m},k) \mid X(1,n,\tilde{m},k)=x] = [1-F(x)]^{m_1-\gamma_1+1}\{x^2[1-F(x)]^{\gamma_2}$$
$$+2\int_x^{b-a} x_2[1-F(x)]^{\gamma_2}dx_2\}.$$

Integrating by parts and using the Binomial expansion we obtain (2.5).

Second, we will prove that (2.5) implies (2.2). We have

$$\gamma_2[1-F(x)]^{m_1-\gamma_1+1}\int_x^{b-a} x_2^2 f(x_2)[1-F(x_2)]^{\gamma_2-1}dx$$

$$= [1-F(x_2)]^{m_1-\gamma_1+1}\left\{x^2[1-F(x)]^{\gamma_2} + 2\sum_{j=0}^{\gamma_2}(-1)^j\binom{\gamma_2}{j}\frac{b(b-a)}{(p_j+1)}\right.$$

$$-2\sum_{j=0}^{\gamma_2}(-1)^j\binom{\gamma_2}{j}\frac{b^2}{(p_j+1)(p_j+2)} - 2\sum_{j=0}^{\gamma_2}(-1)^j\binom{\gamma_2}{j}\frac{x(x+a)^{p_j+1}}{b^{p_j}(p_j+1)}$$

$$\left.+2\sum_{j=0}^{\gamma_2}(-1)^j\binom{\gamma_2}{j}\frac{(x+a)^{p_j+2}}{b^{p_j}(p_j+1)(p_j+2)}\right\}$$

Differentiating both sides *w.r.t.* x and simplifying we get

$$[1-F(x)]^{\gamma_2} = \sum_{j=0}^{\gamma_2}(-1)^j\binom{\gamma_2}{j}\frac{(x+a)^{p_j}}{b^{p_j}}.$$

Therefore

$$F(x)=\frac{1}{b^p}(x+a)^p,$$

This completes the proof.

Weibull distribution

Lemma 2.1: (Haque and Faizan, 2010)

Let $F(x)$ be a *df* such that $F(0)=0$ and has a continuous second order derivative on $(0,\infty)$ with $F'(x)>0$ for all $x>0$ (so that $F(x)<1$ for all x, in particular). If it satisfies the differential equation

$$\frac{\overline{F}''(x)}{\overline{F}(x)}+(\gamma_{r+1}-1)\left[\frac{\overline{F}'(x)}{\overline{F}(x)}\right]^2-\frac{(p-1)}{x}\left[\frac{\overline{F}'(x)}{\overline{F}(x)}\right]$$

$$-\gamma_{r+1}\theta^2 p^2 x^{2(p-1)}=0\,, \tag{2.6}$$

then $\overline{F}(x)=e^{-\theta x^p}$ for all $x>0$ where θ, p, γ_{r+1} are all positive constants.

Proof: Let $\dfrac{\overline{F}'(x)}{\overline{F}(x)}=p\gamma_{r+1}\dfrac{x^{p-1}}{t}$, then (2.6) reduces to

$$\frac{dt}{dx}=px^{p-1}[\gamma_{r+1}^2-\theta^2t^2] \tag{2.7}$$

Therefore,

$$\frac{1}{2\gamma_{r+1}}\int\left[\frac{1}{(\gamma_{r+1}-\theta t)}+\frac{1}{(\gamma_{r+1}+\theta t)}\right]dt=p\int x^{p-1}dx$$

implying that

$$\frac{\gamma_{r+1}+\theta t}{\gamma_{r+1}-\theta t} = A\, e^{2\gamma_{r+1}\theta x^p},$$

where A is the constant of integration.

Thus,

$$\frac{\overline{F}'(x)}{\overline{F}(x)} = \frac{1}{2\gamma_{r+1}}\left[\frac{2A}{Au-1} - \frac{1}{u}\right]\frac{du}{dx}, \text{ where } \quad u = e^{2\gamma_{r+1}\theta x^p}$$

and

$$\overline{F}(x) = B\left[\; Ae^{\gamma_{r+1}\theta x^p} - e^{-\gamma_{r+1}\theta x^p}\;\right]^{1/\gamma_{r+1}} \tag{2.8}$$

where A and B are constants to be determined. Since $F(x)$ is bounded, hence $\overline{F}(x) = e^{-\theta x^p}$ in view of the initial conditions on x.

Theorem 2.3: (Haque and Faizan, 2010)

Let X be a continuous random variable with the *df* $F(x)$ and the *pdf* $f(x)$ over the support $(0,\infty)$. Let $0 < p < \infty$ and $F(x)$ has moment of order $2p$ then for $0 < r < n$,

$$V[X^p(r+1,n,m,k) \mid X(r,n,m,k) = x] = \frac{1}{\gamma_{r+1}^2\theta^2}$$

if and only if

$$\overline{F}(x) = e^{-\theta x^p} \text{ for } x \ge 0 \text{ and } \theta > 0.$$

Proof: It is easy to see that

$$E[X^{pk}(r+1,n,m,k) \mid X(r,n,m,k) = x]$$

$$= \frac{\gamma_{r+1}}{[\overline{F}(x)]^{\gamma_{r+1}}}\int_x^\infty y^{pk}[\overline{F}(y)]^{\gamma_{r+1}-1} f(y)\,dy. \tag{2.9}$$

Thus, for the Weibull distribution

$$\overline{F}(x) = e^{-\theta x^p}, \; x \ge 0, \theta > 0$$

$$E[X^{pk}(r+1,n,m,k) \mid X(r,n,m,k)=x]$$

$$= \frac{\gamma_{r+1}\theta p}{e^{-\gamma_{r+1}\theta x^p}} \int_x^\infty y^{pk} y^{p-1} e^{-\gamma_{r+1}\theta y^p}\, dy$$

$$= \sum_{m=0}^{k} \frac{k!}{m!} x^{pm} a^{p(k-m)}, \text{ where } a^{-p} = \gamma_{r+1}\theta \; .$$

Therefore,

$$V[X^p(r+1,n,m,k) \mid X(r,n,m,k)=x] = a^{2p} = \frac{1}{\gamma_{r+1}^2\theta^2} = c$$

This proves the necessary part.

For sufficiency part, we have

$$\frac{\gamma_{r+1}}{[\bar{F}(x)]^{\gamma_{r+1}}} \int_x^\infty y^{2p} [\bar{F}(y)]^{\gamma_{r+1}-1} f(y)\,dy - \frac{\gamma_{r+1}^2}{[\bar{F}(x)]^{2\gamma_{r+1}}}$$

$$\times \left[\int_x^\infty y^p [\bar{F}(y)]^{\gamma_{r+1}-1} f(y)\,dy \right]^2 = c$$

That is,

$$\gamma_{r+1} [\bar{F}(x)]^{\gamma_{r+1}} \int_x^\infty y^{2p} [\bar{F}(y)]^{\gamma_{r+1}-1} f(y)\,dy$$

$$- \left[\gamma_{r+1} \int_x^\infty y^p [\bar{F}(y)]^{\gamma_{r+1}-1} f(y)\,dy \right]^2 = c[\bar{F}(x)]^{2\gamma_{r+1}} \qquad (2.10)$$

Differentiating (2.10) *w.r.t.* x and solving, we get

$$\gamma_{r+1}\, p \int_x^\infty y^p\, [\bar{F}(x)]^{\gamma_{r+1}-1} f(y)\,dy$$

$$= c\gamma_{r+1} x^{1-p} [\bar{F}(x)]^{\gamma_{r+1}-1} f(x) + p x^p [\bar{F}(x)]^{\gamma_{r+1}} \qquad (2.11)$$

Differentiate (2.11) again *w.r.t.* x, to get

$$c\gamma_{r+1}\, x^{1-p}(\bar{F}(x))^{\gamma_{r+1}-1}\left[-\bar{F}''(x)-\frac{(\gamma_{r+1}-1)(\bar{F}'(x))^2}{\bar{F}(x)}\right.$$

$$\left.+\frac{(p-1)(\bar{F}'(x))}{x}+\frac{1}{c\gamma_{r+1}}x^{2(p-1)}(\bar{F}(x))\,p^2\right]=0.$$

That is,

$$\frac{\bar{F}''(x)}{\bar{F}(x)}+(\gamma_{r+1}-1)\left[\frac{\bar{F}'(x)}{\bar{F}(x)}\right]^2-\frac{(p-1)}{x}\left[\frac{\bar{F}'(x)}{\bar{F}(x)}\right]$$

$$-\gamma_{r+1}\,\theta^2\,p^2x^{2(p-1)}=0 \tag{2.12}$$

Hence $\bar{F}(x)=e^{-\theta x^p}$ in view of the Lemma 2.1.

At $p=1$, this theorem gives the result for exponential distribution.

Remark 2.1: At $m=0, k=1$ and $\gamma_r=n-r+1$, Theorem 2.3 reduces for order statistics as obtained by Beg and Kirmani (1978) at $p=1$ and Khan and Beg (1987).

Remark 2.2: At $m=-1$ and $\gamma_r=k$, Theorem 2.3 reduces for $k-th$ record statistics.

General form of distributions

(a) $\bar{F}(x)=[ax+b]^c,\ x\in(\alpha,\beta),$

Theorem 2.4: (Khan and Alzaid, 2004)

Let X be absolutely continuous random variable with *df* $F(x)$ and *pdf* $f(x)$ on the support (α,β) where α and β may be finite or infinite. Then for $r<s$

$$E[X(s,n,m,k)\,|\,X(r,n,m,k)=x]=a^*x+b^* \tag{2.13}$$

if and only if

$$\bar{F}(x)=[ax+b]^c\ ,x\in(\alpha,\beta) \tag{2.14}$$

where

$$a^* = \prod_{j=1}^{s-r} \frac{c[k+(m+1)(n-r-j)]}{c[k+(m+1)(n-r-j)]+1} = \prod_{j=1}^{s-r} \frac{c\gamma_j}{(1+c\gamma_j)}$$

$$b^* = -\frac{b}{a}(1-a^*) \tag{2.15}$$

Proof: First 'if' part is proved i.e. (2.14) implies (2.13)

We have

$$\overline{F}(x) = [ax+b]^c$$

$$f(x) = -ac[ax+b]^{c-1}$$

where

$$\overline{F}(x) = 1 - F(x)$$

Since the conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \le r < s \le n$, is given by

$$f_{s|r}(y \mid x) = \frac{c_{s-1}}{(s-r-1)!c_{r-1}}$$

$$\times \frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\overline{F}(y)]^{k+(n-s)(m+1)-1} f(y)}{[\overline{F}(x)]^{k+(n-r-1)(m+1)}}$$

This conditional *pdf* can also be written as

$$f_{s|r}(y \mid x) = \frac{c_{s-1}}{(s-r-1)!c_{r-1}(m+1)^{s-r-1}}\left[1 - \left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{m+1}\right]^{s-r-1}$$

$$\times \left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{k+(n-s)(m+1)-1} \frac{f(y)}{\overline{F}(x)}.$$

We have,

$$E[X(s,n,m,k) \mid X(r,n,m,k)=x] = \frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}$$

$$\times \int_x^{\beta} y\left[1-\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{m+1}\right]^{s-r-1}\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{k+(m+1)(n-s)-1}\frac{f(y)}{\overline{F}(x)}dy. \qquad (2.16)$$

Set $u=\dfrac{\overline{F}(y)}{\overline{F}(x)}=\left[\dfrac{ay+b}{ax+b}\right]^c$, then the RHS reduces to

$$=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_0^1\left[\frac{u^{1/c}(ax+b)-b}{a}\right]$$

$$\times[1-u^{m+1}]^{s-r-1}u^{k+(m+1)(n-s)-1}du.$$

Let $u^{m+1}=t$, then we get,

$$E[X(s,n,m,k) \mid X(r,n,m,k)=x]=a^{*}x+b^{*}$$

where

$$b^{*}=-\frac{b}{a}(1-a^{*})$$

and

$$a^{*}=\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_0^1 t^{\frac{1+c[k+(m+1)(n-s)]}{c(m+1)}-1}$$

$$\times(1-t)^{s-r-1}dt$$

which gives

$$a^{*}=\prod_{j=1}^{s-r}\frac{c[k+(m+1)(n-r-j)]}{c[k+(m+1)(n-r-j)]+1}=\prod_{j=1}^{s-r}\frac{c\gamma_j}{(1+c\gamma_j)}$$

after simplification, Hence the 'if' part.

To prove that (2.13) implies (2.14), we have from (2.16)

$$\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_x^\beta y\left[1-\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{m+1}\right]^{s-r-1}$$

$$\times\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{k+(m+1)(n-s)-1}\frac{f(y)}{\overline{F}(x)}dy=a^*x+b^*.$$

Set $\overline{F}(x)=e^{-v}$ and $\overline{F}(y)=e^{-(u+v)}$ to get,

$$\frac{c_{s-1}}{c_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_0^\infty \overline{F}^{-1}[e^{-(u+v)}][1-e^{-u(m+1)}]^{s-r-1}$$

$$\times e^{-u[k+(m+1)(n-s)]}(du)=a^*\overline{F}^{-1}(e^{-v})+b^*.$$

Further, if we set $G(v)=\overline{F}^{-1}(e^{-v})$, we get,

$$\int_0^\infty G(u+v)\mu(du)=G(v)+\frac{b^*}{a^*} \tag{2.17}$$

where

$$\mu(dv)=\frac{c_{s-1}}{a^*c_{r-1}(s-r-1)!(m+1)^{s-r-1}}[1-e^{-u(m+1)}]^{s-r-1}$$

$$\times e^{-u[k+(m+1)(n-s)]}du \tag{2.18}$$

Equation (2.17) is of the form (2.27) with $c^*=\frac{b^*}{a^*}$ and therefore at $\eta\neq 0$,

$$G(v)=\overline{F}^{-1}(e^{-v})=\gamma+\alpha'(1-e^{\eta v}).$$

Let $z=\gamma+\alpha'(1-e^{\eta v})$

Therefore,

$$\overline{F}(z) = \left[1 - \frac{z-\gamma}{\alpha'}\right]^{-\frac{1}{\eta}}$$

$$= [az+b]^c$$

where

$$a = -\frac{1}{\alpha'}, \quad b = \frac{\alpha'+\gamma}{\alpha'}, \quad c = -\frac{1}{\eta} \tag{2.19}$$

Now to see the relationship between a^*, b^* and a, b, c, we have from (2.28) and (2.18),

$$a^* = \frac{c_{s-1}}{c_{r-1}} \frac{B(s-r, \frac{k-\eta}{m+1} + n - s)}{(s-r-1)!(m+1)^{s-r}}$$

Putting $\eta = -\frac{1}{c}$ and solving, we get,

$$a^* = \prod_{j=1}^{s-r} \frac{c[k+(m+1)(n-r-j)]}{c[k+(m+1)(n-r-j)]+1} = \prod_{j=1}^{s-r} \frac{c\gamma_j}{(1+c\gamma_j)}$$

For b^*, we have from (2.26) at $v = 0$,

$$G(0) + c^* = \int_0^\infty G(u)\mu(du).$$

This in view of (2.27) and (2.18) at $\eta \neq 0$ reduces to,

$$\gamma + \frac{b^*}{a^*} = \frac{\gamma + \alpha'}{a^*} - \alpha'$$

Therefore,

$b^* = -\frac{b}{a}(1-a^*)$ in view of (2.19).

Examples:

(i) Power function distribution

$$\bar{F}(x)=\left(\frac{\nu-x}{\nu-\mu}\right)^{\theta}=\left[-\frac{1}{\nu-\mu}x+\frac{\nu}{\nu-\mu}\right]^{\theta},\ \mu\le x\le \nu$$

$$a=-\frac{1}{\nu-\mu},\quad b=\frac{\nu}{\nu-\mu},\quad c=\theta$$

$$a^{*}=\prod_{i=r+1}^{s}\frac{\theta\gamma_i}{\theta\gamma_i+1}<1,\quad b^{*}=\nu(1-a^{*}).$$

(ii) Pareto distribution

$$\bar{F}(x)=\left(\frac{\mu+\delta}{x+\delta}\right)^{\theta}=\left[\frac{1}{\mu+\delta}x+\frac{\delta}{\mu+\delta}\right]^{-\theta},\ \mu\le x<\infty$$

$$a=-\frac{1}{\mu+\delta},\quad b=\frac{\delta}{\mu+\delta},\quad c=-\theta$$

$$a^{*}=\prod_{i=r+1}^{s}\frac{\theta\gamma_i}{\theta\gamma_i+1}>1,\quad b^{*}=\delta(1-a^{*}).$$

(iii) Exponential distribution

$$\bar{F}(x)=e^{-\lambda(x-\mu)},\ x\ge\mu$$

$$=\left[1-\frac{\lambda(x-\mu)}{c}\right]^{c},$$

$$a=-\frac{\lambda}{c},\quad b=\frac{c+\lambda\mu}{c},\quad c\to\infty$$

$$a^{*}=1,\quad b^{*}=\frac{1}{\lambda}\sum_{i=r+1}^{s}\frac{1}{\gamma_i}.$$

Theorem 2.5: (Khan *et al.*, 2006)

Let $\xi(x)$ be a monotonic and continuous function of x. If

$$E[\xi\{X(s,n,m,k)\}|X(r,n,m,k)=x]=g_{s|r}(x) \tag{2.20}$$

then

$$\overline{F}(x)=\exp\left[-\frac{1}{\gamma_{r+1}}\int_{\alpha}^{x}\frac{g'_{s|r}(t)}{[g_{s|r}(t)-g_{s|r+1}(t)]}dt\right] \tag{2.21}$$

Proof: We have,

$$E[\xi\{X(s,n,m,k)\}|X(r,n,m,k)=x]=g_{s|r}(x)$$

that is,

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_{x}^{\beta}\xi(y)[(\overline{F}(x))^{m+1}-(\overline{F}(y))^{m+1}]^{s-r-1}$$

$$\times[\overline{F}(y)]^{\gamma_s-1}f(y)dy=g_{s|r}(x)[\overline{F}(x)]^{\gamma_{r+1}}$$

Differentiating both sides with respect to x and re-arranging the terms, we get

$$\frac{f(x)}{\overline{F}(x)}=-\frac{1}{\gamma_{r+1}}\frac{g'_{s|r}(x)}{[g_{s|r+1}(x)-g_{s|r}(x)]}$$

and hence the Theorem.

The result for $s=r+1$, obtained by Raqab and Abu-Lawi (2004) and Cramer *et al.* (2004), may be deduced by noting that

$$g_{r+1|r+1}(x)=E[\xi\{X(r+1,n,m,k)\}|X(r+1,n,m,k)=x]=\xi(x).$$

Corollary 2.1: (Khan *et al.*, 2006)

If $E[\{X(s,n,m,k)\}|X(r,n,m,k)=x]=a^{*}_{s|r}x+b^{*}_{s|r}=g_{s|r}(x)$

then

$$\overline{F}(x)=[ax+b]^{c},$$

where $a^*_{s|r} = \prod_{j=r+1}^{s} \frac{c\gamma_j}{1+c\gamma_j}$, $b^*_{s|r} = -\frac{b}{a}(1-a^*_{s|r})$

Proof: The proof was given by Khan and Alzaid (2004) using Rao and Shanbhag (1994) result. This can also be obtained from Theorem 2.5 under the assumption $[a\alpha+b]^c = 1$ by noting that

$$g_{s|r+1}(x) - g_{s|r}(x) = a^*_{s|r+1}x + b^*_{s|r+1} - a^*_{s|r}x - b^*_{s|r}$$

$$= (a^*_{s|r+1} - a^*_{s|r})\left(x + \frac{b}{a}\right)$$

$$= \frac{a^*_{s|r}}{ac\gamma_{r+1}}(ax+b)$$

as

$$a^*_{s|r+1} = \prod_{j=r+2}^{s} \frac{c\gamma_j}{1+c\gamma_j} = \frac{1+c\gamma_{r+1}}{c\gamma_{r+1}} a^*_{s|r}$$

Therefore,

$$\frac{g'_{s|r}(x)}{\gamma_{r+1}[g_{s|r+1}(x) - g_{s|r}(x)]} = \frac{ac}{ax+b}$$

Thus,

$$\frac{f(x)}{\overline{F}(x)} = -\frac{ac}{ax+b}$$

implying

$$\overline{F}(x) = [ax+b]^c$$

For $s = r+1$, the result was obtained by Keseling (1999).

Further, it has been shown by Bieniek and Szynal (2003), Khan and Alzaid (2004) and Cramer *et al.* (2004), that for

(i) Power function distribution

$$\bar{F}(x) = \left(\frac{\nu - x}{\nu - \mu}\right)^{\theta}, \quad \mu < x < \nu \tag{2.22}$$

$$a^{*}_{s|r} = \prod_{i=r+1}^{s} \frac{\theta\gamma_i}{1+\theta\gamma_i} < 1, \quad b^{*}_{s|r} = \nu(1 - a^{*}_{s|r})$$

(ii) Pareto distribution

$$\bar{F}(x) = \left(\frac{\mu + \delta}{x + \delta}\right)^{\theta}, \quad \mu < x < \infty \tag{2.23}$$

$$a^{*}_{s|r} = \prod_{i=r+1}^{s} \frac{\theta\gamma_i}{\theta\gamma_i - 1} > 1, \quad b^{*}_{s|r} = \delta(a^{*}_{s|r} - 1)$$

(iii) Exponential distribution

$$\bar{F}(x) = e^{-\lambda(x-\mu)}, \quad x \geq \mu \tag{2.24}$$

$$a^{*}_{s|r} = 1, \quad b^{*}_{s|r} = \frac{1}{\lambda}\sum_{i=r+1}^{s} \frac{1}{\gamma_i}$$

Remark 2.3: Let $\xi(x)$ be a monotonic and continuous function of x, then it has been shown by Khan and Alzaid (2004) that

$$E[\xi\{X(s,n,m,k)\} \mid X(r,n,m,k) = x] = a^{*}_{s|r}\,\xi(x) + b^{*}_{s|r}$$

if and only if $\bar{F}(x) = [a\xi(x) + b]^{c}$

This can be deduced from Theorem 2.5 by considering

$$g_{s|r}(x) = a^{*}_{s|r}\,\xi(x) + b^{*}_{s|r}$$

A number of distributions can be characterized by the proper choice of a, b, c and $\xi(x)$; (Khan and Alzaid, 2004):

Remark 2.4: For order statistics $(m = 0, k = 1)$ characterizing results based on linear regression were obtained by Khan and Ali (1987), Khan and Abu-Salih

(1989), Franco and Ruiz (1995, 1997), López-Blázquez and Moreno-Rebollo (1997), Wesolowski and Ahsanullah (1997) and Dembińska and Wesolowski (1998) and Khan and Abouammoh (2000).

Remark 2.5: For record statistics $(m=-1, k=1)$ distributions were characterized using conditional expectations by Nagaraja (1988), Franco and Ruiz (1996, 1997), Ahsanullah and Wesolowski (1998), Dembińska and Wesolowski (2000) and Athar *et al.* (2003).

Theorem 2.6: (Samuel, 2008)

Let X be a random variable variable with absolutely continuous *cdf* $F(x)$ and *pdf* $f(x)$. Suppose $F(x)<1$ for all $x \in (\alpha,\beta)$, $F(\alpha)=0$ and $F(\beta)=1$. Then

$$F(x)=1-[ah(x)+b]^c \text{ for } x \in (\alpha,\beta) \tag{2.25}$$

if and only if for $r<n$,

$$E[h(X(r+1,n,m,k)) \mid X(r+1,n,m,k)=x]$$

$$=\frac{ac[k+(n-r+-)(m+1)]h(x)-b}{a\{c[k+(n-r-1)(m+1)]+1\}}, \tag{2.26}$$

where $h(\cdot)$ is a monotonic, continuous and differentiable function on (α,β), $a \neq 0$ and $c[k+(n-r-1)(m+1)]+1 \neq 0$.

Proof: Suppose that equation (2.25) holds good. Then

$$1-F(x)=[ah(x)+b]^c=-\frac{ah(x)+b}{ach'(x)}f(x) \tag{2.27}$$

Using (4.4) we write the following.

$$E[h(X(r+1,n,m,k)) \mid X(r,n,m,k)=x]=[1-F(x)]^{m-\gamma_r+1}I_1 \tag{2.28}$$

where,

$$I_1=\gamma_{r+1}\int_x^{F^{-1}(1)} h(y)[1-F(y)]^{\gamma_{r+1}-1}f(y)dy \tag{2.29}$$

By applying integration by parts to the integral on the right side of (2.29) treating $h(y)$ as first function and the rest as second function, we get

$$I_1 = h(x)[1-F(x)]^{\gamma_r+1} + \int_x^{F^{-1}(1)} h'(y)[1-F(y)]^{\gamma_r+1} dy \qquad (2.30)$$

Now using the expression for $1-F(\cdot)$ given by (2.27) in the integrand of the integral on the right side of (2.30) and simplifying, we obtain

$$I_1 = \frac{ac\gamma_{r+1}h(x)-b}{a(c\gamma_{r+1}+1)}[1-F(x)]^{\gamma_{r+1}} \qquad (2.31)$$

Then the result (2.26) follows from equations (2.28) and (2.31).

Conversely, we assume that equation (2.26) holds good. Then we have from (2.28),

$$I_1 = \frac{E[h(X(r+1,n,m,k)) \mid X(r,n,m,k)=x]}{[1-F(x)]^{m-\gamma_r+1}}$$

That is,

$$\gamma_{r+1}\int_x^{F^{-1}(1)} h(y)[1-F(y)]^{\gamma_{r+1}-1} f(y)dy$$

$$= \frac{ac\gamma_{r+1}h(x)-b}{a(c\gamma_{r+1}+1)}[1-F(x)]^{\gamma_r-m-1} \qquad (2.32)$$

If we differentiate both sides of (2.32) *w.r.t.* x and then simplify the resulting equation, we get

$$1-F(x) = -\frac{ah(x)+b}{ach'(x)}f(x) \qquad (2.33)$$

Then equation (2.25) follows from (2.27) and (2.33). Hence the theorem is proved.

Remark 2.6: If we put $k=1$ and $m=0$ in equation (2.26) of Theorem 2.6, we get the existing Theorem 2.1 of Khan and Abu-Salih (1989).

Remark 2.7: By the proper choice of a, b, c and $h(x)$, Theorem 2.6 characterizes several well known continuous probability distributions by conditional expectation of generalized order statistics. For example, if we choose $a = a^p$, $b = 0$, $c = 1$ and $h(x) = x^{-p}$ then Pareto distribution is characterized. For the choice of a, b, c and $h(x)$ for other distributions, see Khan and Abu-Salih (1989).

3. CHARACTERIZATION THROUGH DISTRIBUTIONAL PROPERTY

Exponential distribution

Theorem 3.1: (Kamps and Gather, 1997)

Let $F(x)$ be absolutely continuous with density function $f(x)$, $F(0) = 0$, and suppose that $F(x)$ is strictly increasing on $(0, \infty)$, and either is new better than used (NBU) or new worse than used (NWU). Moreover, let the expected values involved be finite. Then $F(x) \equiv \exp(\lambda)$ for some $\lambda > 0$, if and only if there exist integers r and n, $1 \le r \le n-1$, such that

$$E[X(r+1,n,m,k)] - E[X(r,n,m,k)] \underset{=}{d} E[X(1,n-r,m,k)].$$

Proof: Making use of the representations

$$E[X(r+1,n,m,k) - X(r,n,m,k)]$$

$$= \int_0^\infty \int_x^\infty f^{W_{r,r+1,n}}(w)dwdx$$

$$= \frac{c_r}{(r-1)!} \int_0^\infty \int_x^\infty \int_0^\infty [1-F(x)]^m f(y) g_m^{r-1}[F(y)]$$

$$\times [1-F(y+w)]^{\gamma_{r+1}-1} f(y+w)dydwdx$$

$$= \frac{c_r}{(r-1)!\gamma_{r+1}} \int_0^\infty \int_0^\infty [1-F(y)]^m f(y) g_m^{r-1}(F(y))$$

$$\times [1-F(x+y)]^{\gamma_{r+1}} dydx$$

$$= \int_0^\infty \int_0^\infty f^{X(r,n,m,k)}(y) \left[\frac{[1-F(x+y)]^{\gamma_{r+1}}}{[1-F(y)]^{\gamma_{r+1}}} \right] dydx$$

and

$$E[X(1,n-r,m,k)]$$

$$=\gamma_{r+1}\int_0^\infty\int_0^\infty[1-F(y)]^{\gamma_{r+1}-1}f(y)dydx$$

$$=\int_0^\infty[1-F(x)]^{\gamma_{r+1}}dx$$

$$=\int_0^\infty\int_0^\infty f^{X(r,n,m,k)}(y)[1-F(x)]^{\gamma_{r+1}}dydx$$

we obtain,

$$E[W_{r,r+1:n}]=E[X(1,n-r,m,k)]$$

$$\Leftrightarrow\int_0^\infty\int_0^\infty f^{X(r,n,m,k)}(y)[1-F(x+y)]^{\gamma_{r+1}}/[1-F(y)]^{\gamma_{r+1}}$$

$$-[1-F(x)]^{\gamma_{r+1}}]dydx=0,$$

which implies the assertion.

Theorem 3.2: (Ahsanullah, 2000)

Let X be a non-negative random variable having an absolutely continuous (with respect to Lebesgue measure) strictly increasing distribution function $F(x)$ with $F(0)=0$ and $F(x)<1$ for all $x>0$. Then the following properties are equivalent:

(a) X has an exponential distribution with density as given,

$$f(x,\theta)=\begin{cases}\theta\exp(-\theta x), & x>0,\ \theta>0,\\ 0, & otherwise.\end{cases}\tag{3.1}$$

(b) For $1<r\le n$, the statistics

$\{k+(n-r)(m+1)\}\{X(r,n,m,k)-\{X(r-1),n,m,k\}$ and $\{X(r-1),n,m,k\}$ are independent.

Proof: Integrating out $x_1,\ldots\ldots,x_{r-2},x_{r+1},\ldots\ldots,x_n$, and using the transformation,

$$U=X(r-1,n,m,k)\text{ and }W=\gamma_r\{X(r,n,m,k)-X(r-1,n,m,k)\},$$

we get on simplification the *joint pdf* of $f_{UW}(u,w)$ of U and W as

$$f_{UW}(u,w)=\begin{cases}\dfrac{c_{r-2}}{(r-2)!}[1-F(u)]^m g_m^{r-2}[F(u)]\left[1-F\left(u+\dfrac{w}{\gamma_r}\right)\right]^{\gamma_r-1} \\ \qquad\qquad\qquad\qquad \times f(u)f\left(u+\dfrac{w}{\gamma_r}\right),\ 0<u,w<\infty, \\ 0,\ otherwise.\end{cases} \tag{3.2}$$

If X has the *joint pdf* as given in equation (3.1), then

$$\overline{F}(x)=1-F(x)=e^{-\theta x}$$

and

$$g_m(F(x))=\begin{cases}\dfrac{1}{m+1}(1-e^{-(m+1)\theta x}),\ m\neq -1 \\ \theta x, \qquad\qquad m=-1.\end{cases}$$

on simplification, we get from equation (3.2), the *joint pdf* $f_{UW}(u,w)$ of U and W for the exponential distribution as

$$f_{UW}(u,w)=\frac{c_{r-2}\theta^2}{(r-2)!}g_m^{r-2}(1-e^{-\theta u})e^{-\gamma_{r-1}\theta u}e^{-\theta w},\ 0<u,w<\infty. \tag{3.3}$$

Thus W and U are independently distributed.

Let $f_U(u)$ be the *pdf* of $U=X(r-1,n,m,k)$, then

$$f_U(u)=\frac{c_{r-2}}{(r-2)!}[1-F(u)]^{-1+\gamma_{r-1}}g_m^{r-2}[F(u)]f(u) \tag{3.4}$$

Using equation (3.2) and (3.3) and the relation $\gamma_{r-1}=\gamma_r+m+1$, we get the conditional distribution of $f_{W|U}(W\mid U=u)$ as

$$f_{W|U}(W \mid U=u)=\left[\bar{F}\left(u+\frac{w}{\gamma_r}\right)(\bar{F}(u))^{-1}\right]^{\gamma_{r-1}} f\left(u+\frac{w}{\gamma_r}\right)[\bar{F}(u)]^{-1},$$

for all w_1, $0 < w_1 < \infty$ (3.5)

and all u.

Integrating the expression in equation (3.5) with respect to w from w_1 to ∞, we get

$$\bar{F}_{W|w}(w_1 \mid U=u)=\left[\bar{F}\left(u+\frac{w_1}{\gamma_r}\right)[\bar{F}(u)]^{-1}\right]^{\gamma_r},$$

for all w_1 and $0 < u, w_1 < \infty$. (3.6)

Since W and U are independent, we get from equation (3.6) for all w and u.

$$\left[\bar{F}\left(u+\frac{w_1}{\gamma_r}\right)[\bar{F}(u)]^{-1}\right]^{\gamma_r}=G(w),$$

where $G(w)$ is a function of w only. Now taking the limit $u \to 0$. we get

$$G(w)=\left[\bar{F}\left(\frac{w}{\gamma_r}\right)\right]^{\gamma_r}.$$

Hence for all w and u, $0 < u, w < \infty$ and a fixed γ_r, we get

$$\bar{F}\left(u+\frac{w}{\gamma_r}\right)=\bar{F}(u)\bar{F}\left(\frac{w}{\gamma_r}\right). \qquad (3.7)$$

The non-degenerate solution of equation (3.7) with the boundary conditions $\bar{F}(0)=1$ and $\bar{F}(\infty)=0$ is $\bar{F}(u)=e^{-\theta u}$, $\theta > 0$ and $0 \le u < \infty$.

Remark 3.1: If $k=1$ and $m=0$, then from the Theorem 3.2, we obtain the result of Rossberg (1972). If $k=1$ and $m=-1$, then we obtain from Theorem 3.2, the characterization result of the exponential distribution given by

Ahsanullah (1978) based on the independence of $X_{U(n)} - X_{U(n-1)}$ and $X_{U(n-1)}$.

Lemma 3.1: (Ahsanullah, 2006)

$$F_{r,n,m,k}(x) = I_{\alpha(x)}(r, \frac{\lambda_r}{m+1}) \text{ if } m > -1 \tag{3.8}$$

and

$$F_{r,n,m,k}(x) = \Gamma\beta(x)(r) \text{ if } m = -1 \tag{3.9}$$

where,

$$\alpha(x) = 1 - [\overline{F}(x)]^{m+1}, \ \overline{F}(x) = 1 - F(x)$$

$$\beta(x) = -k \ln \overline{F}(x),,$$

$$I_x(p,q) = \frac{1}{B(p,q)} \int_0^x u^{p-1}(1-u)^{q-1} du, \ B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

$$\Gamma_x(r) = \int_0^\infty \frac{1}{\Gamma(r)} u^{r-1} e^{-u} du.$$

Proof: For $m > -1$, from (3.8)

$$F_{r,n,m,k}(x) = \frac{C_r}{(r-1)!} \int_{F^{-1}(0)}^x [1 - F(u)]^{k+(n-r)(m+1)-1}$$

$$\times [g_m^{r-1}(F(u)] f(u) du$$

Using the relation $B(r, \frac{\gamma_r}{m+1}) = \frac{\Gamma(r)(m+1)^r}{c_r}$ and substituting

$t = 1 - [\overline{F}(x)]^{m+1}$, we get on simplification

$$F_{r,n,m,k}(x) = \frac{1}{B(r, \frac{\gamma_r}{m+1})} \int_0^{1-[\overline{F}(x)]^{m+1}} (1-u)^{\gamma_r - 1}(1-u)^{r-1} du$$

$$= I_{\alpha(x)}(r, \frac{\lambda_r}{m+1})$$

For $m=-1$

$$F_{r,n,m,k}(x)=\frac{k^r}{(r-1)!}\int_{F^{-1}(0)}^{x}[1-F(u)]^{k-1}$$

$$\times[-\ln(1-F(u))]^{r-1}f(u)du$$

$$=\frac{1}{(r-1)!}\int_0^{-k\ln\overline{F}(x)}t^{r-1}e^{-t}dt$$

$$=\Gamma\beta(x)(r),\quad \beta(x)=-k\ln\overline{F}(x).$$

Lemma 3.2: (Ahsanullah, 2006)

$$F_{r,n,m,k}(r,\frac{\lambda_r}{m+1})-F_{r,n,m,k}(r+1,\frac{\lambda_{r+1}}{m+1})$$

$$=\frac{1}{\gamma_{r+1}}\frac{\overline{F}(x)}{f(x)}f_{r+1,n,m,k}(x)$$

Proof: For $m>-1$,

$$F_{r,n,m,k}(r,\frac{\lambda_r}{m+1})-F_{r,n,m,k}(r+1,\frac{\lambda_{r+1}}{m+1})$$

$$=I_{\alpha(x)}(r,\frac{\lambda_r}{m+1})-I_{\alpha(x)}(r+1,\frac{\gamma_{r+1}}{m+1}).$$

Using the relation

$$I_x(a,b)-I_x(a+1,b-1)=\frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)}x^a(1-x)^{b-1},$$

we obtain

$$I_{\alpha(x)}(r,\frac{\gamma_r}{m+1})-I_{\alpha(x)}(r+1,\frac{\gamma_{r+1}}{m+1}-1)$$

$$=\frac{\Gamma(r+\frac{\gamma_r}{m+1})}{\Gamma(r+1)\Gamma(\frac{\gamma_r}{m+1})}[\alpha(x)]^r[1-\alpha(x)]^{\frac{\gamma_r}{m+1}-1}. \tag{3.10}$$

We know that

$$B(r,\frac{\gamma_r}{m+1})=\frac{\Gamma(r)(m+1)^r}{c_r} \text{ and hence}$$

$$\frac{\Gamma(r+\frac{\gamma_r}{m+1})}{\Gamma(r+1)\Gamma(\frac{\gamma_r}{m+1})}[\alpha(x)]^r[1-\alpha(x)]^{\frac{\gamma_r}{m+1}-1}$$

Thus

$$F_{r,n,m,k}(r,\frac{\lambda_r}{m+1})-F_{r,n,m,k}(r+1,\frac{\lambda_{r+1}}{m+1})$$

$$=\frac{1}{\gamma_{r+1}}\frac{\overline{F}(x)}{f(x)}f_{r+1,n,m,k} \tag{3.11}$$

For $m=-1$,

$$F_{r,n,m,k}(r,\frac{\lambda_r}{m+1})-F_{r,n,m,k}(r+1,\frac{\lambda_{r+1}}{m+1})$$

$$=\Gamma_{\beta(x)}(x)-\Gamma_{\beta(x)}(r+1) \tag{3.12}$$

Using the relation

$$\Gamma_{\beta(x)}(r)-\Gamma_{\beta(x)}(r+1)=\frac{[\beta(x)]^r e^{-\beta(x)}}{\Gamma(r+1)}$$

$$=\frac{[-k\ln\overline{F}(x)]^r e^{k\ln\overline{F}(x)}}{\Gamma(r+1)}$$

$$=\frac{[-k\ln\overline{F}(x)]^r[\overline{F}(x)]^k}{\Gamma(r+1)},$$

we obtain

$$F_{r,n,m,k}(r,\frac{\lambda_r}{m+1})-F_{r,n,m,k}(r+1,\frac{\lambda_{r+1}}{m+1})=\frac{1}{k}\frac{\overline{F}(x)}{f(x)}f_{r+1,n,m,k}(x).$$

Theorem 3.3: (Ahsanullah, 2006)

Let X be a non-negative random variable having an absolutely continuous (with respect to Lebesgue measure) distribution function $F(x)$ with $F(x)=0$ and $0<F(x)<1$ for all $x>0$. Then the following properties are equivalent

(a) X has an exponential distribution with density as $f(x)=\sigma^{-1}e^{-\sigma^{-1}x}$

(b) $X(r+1,n,m,k) \overset{d}{=} X(r,n,m,k)+\sigma\dfrac{W}{\gamma_{r+1}}$, $r>1$, where W is independent of $X(r+1,n,m,k)$ and $X(r,n,m,k)$ and $W\in E(1)$ and $m\geq -1$.

Proof: The proof of (a) implies (b) follows from

$$X(r+1,n,m,k) \overset{d}{=} X(r,n,m,k)+\sigma\frac{W}{\gamma_{r+1}}.$$

We will prove here (b) implies (a).

Using the relation

$$X(r+1,n,m,k) \overset{d}{=} X(r,n,m,k)+\sigma\frac{W}{\gamma_{r+1}},$$

we obtain for $m>-1$,

$$F_{r+1,n,m,k}(x)=\int_0^x (1-e^{-\frac{\gamma_{r+1}}{\sigma}(x-y)})f_{r,n,m,k}(y)dy$$

$$=F_{r,n,m,k}(x)-\int_0^x (1-e^{-\frac{\gamma_{r+1}}{\sigma}(x-y)})f_{r,n,m,k}(y)dy \quad (3.13)$$

Thus

$$e^{\frac{\gamma_{r+1}}{\sigma}x}[F_{r,n,m,k}(x)-F_{r+1,n,m,k}(x)]=\int_0^x e^{-\frac{\gamma_{r+1}}{\sigma}(x-y)}$$

$$\times f_{r,n,m,k}(y)dy$$

Differentiating both sides of the above equation, we obtain

$$e^{\frac{\gamma_{r+1}}{\sigma}x}[f_{r,n,m,k}(x)-f_{r+1,n,m,k}(x)]+\frac{\gamma_{r+1}}{\sigma}e^{\frac{\gamma_{r+1}}{\sigma}x}[F_{r,n,m,k}(x)-F_{r+1,n,m,k}(x)]$$
$$=e^{\frac{\gamma_{r+1}}{\sigma}x}f_{r,n,m,k}(x)$$

On simplification, we have

$$\frac{\gamma_{r+1}}{\sigma}[F_{r,n,m,k}(x)-F_{r+1,n,m,k}(x)]=f_{r+1,n,m,k}(x)$$

By Lemma 3.2

$$\frac{1}{\sigma}\frac{\overline{F}(x)}{f(x)}f_{r+1,n,m,k}(x)=f_{r+1,n,m,k}(x) \tag{3.14}$$

Thus

$$\frac{f(x)}{\overline{F}(x)}=\frac{1}{\sigma} \tag{3.15}$$

Using the boundary condition $F(0)=0$ and $F(x)<1$ for all $x>0$. We must have

$$F(x)=1-e^{-x\sigma^{-1}},\ x>0,\ \sigma>0. \tag{3.16}$$

For $m=-1$,

$$F_{r+1,n,m,k}(x)=\int_0^x(1-e^{-\frac{k}{\sigma}(x-y)})f_{r,n,m,k}(y)dy \tag{3.17}$$

i.e.

$$e^{\frac{k}{\sigma}x}[F_{r,n,m,k}(x)-F_{r+1,n,m,k}(x)]=\int_0^x e^{\frac{k}{\sigma}y}f_{r,n,m,k}(y)dy \tag{3.18}$$

Differentiating both sides of the above equation with respect to x, we obtain

$$e^{\frac{k}{\sigma}x}[f_{r,n,m,k}(x)-f_{r+1,n,m,k}(x)]+\frac{k}{\sigma}e^{\frac{k}{\sigma}x}[F_{r,n,m,k}(x)-F_{r+1,n,m,k}(x)]$$
$$=e^{\frac{k}{\sigma}x}f_{r,n,m,k}(x)$$

On simplification, we obtain

$$\frac{k}{\sigma}[F_{r,n,m,k}(x)-F_{r+1,n,m,k}(x)]=f_{r+1,n,m,k}(x)$$

i.e.

$$\frac{1}{\sigma}\frac{\overline{F}(x)}{f(x)}f_{r+1,n,m,k}(x)=f_{r+1,n,m,k}(x)$$

Thus

$$\frac{f(x)}{\overline{F}(x)}=\frac{1}{\sigma} \tag{3.19}$$

Using the boundary condition $F(0)=0$ and $F(x)<1$ for all $x>0$. We must have

$$F(x)=1-e^{-x\sigma^{-1}},\ x>0,\ \sigma>0. \tag{3.20}$$

For $k=1$ and $m=0$, we obtain a characterization of the exponential distribution by the relation $X_{r+1,n}\overset{d}{=}X_{r,n}+\sigma\frac{W}{n-r}$. If $k=1$ and $m=-1$,

Then we obtain a characterization of the exponential distribution by the relation $X_{U(r+1)}\overset{d}{=}X_{U(r)}+\sigma W$, where $X_{U(i)}$ is the i^{th} upper record in a sequence, $\{X_n, n\geq 1\}$, of *i.i.d.* observations.

Generalized Pareto distribution

Let X be a lifetime (non-negative) random variable with distribution function $F(x)$ and survival function $\overline{F}(x)=1-F(x)$. The random variable X is said to have generalized Pareto distribution (*GPD*) if its survival function is given by

$$\overline{F}(x) = \left(\frac{b}{ax+b}\right)^{\frac{1}{a}+1}, \ x \geq 0, \tag{3.21}$$

where $a > -1$ and $b > 0$.

Theorem 3.4: (Tavangar and Asadi, 2007)

Let $X(1,n,m,k),...,X(n,n,m,k)$ be non-negative generalized order statistics from a population with absolutely continuous distribution function $F(x)$. Furthermore, assume that

$$\theta(t) = \frac{m(t)}{m(0)},$$

where m denotes the mean residual life function of $F(x)$. Then $F(x)$ is *GPD* if and only if for $t > 0$, and $s = 1,2,...,n$,

$$\left.\frac{X(s,n.m.k)-t}{\theta(t)}\right| X(1,n,m,k) > t \ \underline{\underline{d}} \ X(s,n,m,k), \tag{3.22}$$

where d stands for distribution.

Proof: We prove first the 'if' part of the Theorem. First note that for $u > 0$,

$$P(X(1,n,m,k) > u) = \int_u^\infty [k+(m+1)(n-1)][(\overline{F}(x)]^{(m+1)(n-1)+k-1} dF(x)$$

$$= [\overline{F}(u)]^{(m+1)(n-1)+k} \tag{3.23}$$

Let relation (3.22) hold. If $s = 1$, then we have

$$P\left(\frac{X(1,n.m.k)-t}{\theta(t)} > u \middle| X(1,n,m,k) > t\right) = P(X(1,n,m,k) > u)$$

or,

$$P(X(1,n,m,k) > t + u\theta(t)) = P(X(1,n,m,k) > t)P(X(1,n,m,k) > u).$$

This implies, using (3.23), that for all $t > 0$ and $u > 0$,

$$\overline{F}(t + u\theta(t)) = \overline{F}(t)\overline{F}(u).$$

That is, $F(x)$ is *GPD*. Now suppose that $2 \le s \le n$. Let

$$G(u,t) = \frac{P(t < X(1,n,m,k) < X(s,n,m,k) \le t + u\theta(t))}{P(X(1,n,m,k) > t)}.$$

Then,

$$G(u,t) = \frac{1}{[\bar{F}(t)]^{(m+1)(n-1)+k}}$$

$$\times \int_t^{t+u\theta(t)} \int_t^y \frac{c_{s-1}}{(s-2)!} \bar{F}^m(x)\{h_m(F(y)) - h_m(F(x))\}^{s-2}$$

$$\times [\bar{F}(y)]^{(m+1)(n-s)+k-1} dF(x) dF(y)$$

$$= \frac{1}{[\bar{F}(t)]^{(m+1)(n-1)+k}} \int_t^{t+u\theta(t)} \frac{c_{s-1}}{(s-1)!} \{h_m(F(y)) - h_m(F(t))\}^{s-1}$$

$$\times [\bar{F}(y)]^{(m+1)(n-s)+k-1} dF(y)$$

$$= \int_t^{t+u\theta(t)} \frac{c_{s-1}}{(s-1)!} \left\{ \frac{h_m(F(y)) - h_m(F(t))}{\bar{F}^{m+1}(t)} \right\}^{s-1}$$

$$\times \left(\frac{\bar{F}(y)}{\bar{F}(t)} \right)^{(m+1)(n-s)+k-1} \frac{dF(y)}{\bar{F}(t)}$$

$$= \int_o^{1-[\bar{F}(t+u\theta(t))/\bar{F}(t)]} \frac{c_{s-1}}{(s-1)!} \left\{ \frac{h_m(1-(1-v)\bar{F}(t)) - h_m(F(t))}{\bar{F}^{m+1}(t)} \right\}^{s-1}$$

$$\times (1-v)^{(m+1)(n-s)+k-1} dv. \tag{3.24}$$

Note that,

$$\frac{dh_m(x)}{dx} = (1-x)^m$$

One can easily show that for each $m \in R$, the fraction

$$\frac{h_m(1-(1-v)\bar{F}(t)) - h_m(F(t))}{\bar{F}^{m+1}(t)}$$

does not depend on t and is only a function of v. In fact, it is the same as $g_m(v)$.

On the other hand,

$$P(X(s,n,m,k) \le u) = \int_0^u \frac{c_{s-1}}{(s-1)!}[\overline{F}(x)]^{(m+1)(n-s)+k-1} g_m^{s-1}[F(x)]dF(x)$$

$$= \int_0^{1-\overline{F}(u)} \frac{c_{s-1}}{(s-1)!} g_m^{s-1}(v)(1-v)^{(m+1)(n-s)+k-1} dv \quad (3.25)$$

If (3.22) holds then we have

$$G(u,t) = P(X(s,n,m,k) \le u) \quad (3.26)$$

This implies that the right-hand sides of (3.24) and (3.25) are equal which in turn implies that for every t, $u > 0$,

$$\frac{\overline{F}(t + u\theta(t))}{\overline{F}(t)} = \overline{F}(u) \quad (3.27)$$

That is, $F(x)$ is *GPD*.

The 'only if' part of Theorem follows equation (3.24) and (3.25) and fact that when X is distributed as *GPD*, equation (3.27) holds. This completes the proof of the theorem.

Theorem 3.5: (Tavangar and Asadi, 2007)

Let $X_1, X_2, \ldots, X_n$ be *i.i.d* non-negative random variables with absolutely continuous distribution function $F(x)$. Assume that $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote order statistics corresponding to X_i's, $i = 1,2,\ldots,n$. Then for fixed k and n, where $k = 1,2,\ldots,n-2$,

$$E\left(\frac{X_{k+1:n} - X_{k:n}}{\theta(X_{k:n})} \middle| X_{k:n} > t\right) = c, \quad (3.28)$$

if and only if $F(x)$ is *GPD*, where c is a positive constant.

Proof: See reference.

Uniform distribution

Theorem 3.6: (Tavangar and Asadi, 2007)

Let $X_1, X_2, \ldots, X_n$ be independent non-negative random variables from a continuous distribution function $F(x)$. Let also $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote corresponding order statistics, then

$$t - X_{r:n} \mid X_{n:n} \le t \overset{d}{=} X_{n-r+1:n} \mid X_{n:n} \le t, \; t > 0, \tag{3.29}$$

for some n and $r, r = 1, 2, \ldots, n,$ if and only if for some $\theta > 0$, $F(x)$ is uniform on $(0, \theta)$, where d stands for distribution.

Proof: Let $x < t$. To prove the 'only if ' part of the theorem note that (3.29) is equivalent to

$$P(t - X_{r:n} \ge x \mid X_{n:n} \le t) = P(X_{n-r+1:n} \ge x \mid X_{n:n} \le t).$$

Using the following equation,

$$P(t - X_{r:n} \ge x \mid X_{n:n} \le t) = \frac{P(X_{r:n} \le t - x, X_{n:n} \le t)}{P(X_{n:n} \le t)}$$

$$= \sum_{i=r}^{n} \binom{n}{i} \left(\frac{F(t-x)}{F(t)} \right)^i \left(1 - \frac{F(t-x)}{F(t)} \right)^{n-i},$$

we get for all $0 < x < t$,

$$\sum_{i=r}^{n} \binom{n}{i} (\varphi_t(t-x))^i (1 - \varphi_t(t-x))^{n-i}$$

$$= 1 - \sum_{i=n-r+1}^{n} \binom{n}{i} (\varphi_t(x))^i (1 - \varphi_t(x))^{n-i}$$

$$= 1 - \left\{ 1 - \sum_{i=0}^{n-r} \binom{n}{i} (\varphi_t(x))^i (1 - \varphi_t(x))^{n-i} \right\}$$

$$= \sum_{i=r}^{n} \binom{n}{i} (1 - \varphi_t(x))^i (\varphi_t(x))^{n-i},$$

where,

$$\varphi_r(x) = \frac{F(x)}{F(t)}.$$

This implies that,

$$\frac{F(t-x)}{F(t)} = 1 - \frac{F(x)}{F(t)},$$

or,

$$F(u+v) = F(u) + F(v), \quad v > 0, u > 0.$$

The only solution of this functional equation is

$$F(x) = cx, \, c > 0.$$

Since $F(x)$ is a distribution function it must have a finite right extremely, say θ. Hence, $F(x)$ is uniform on $(0, \theta)$.

The 'if' part of the Theorem is straightforward.

Theorem 3.7: (Tavangar and Asadi, 2007)

Let $X_1, X_2, ..., X_n$ be independent non-negative random variables from a continuous distribution function $F(x)$, such that $\lim F(t)/t$ exists as $t \to 0$. Then for given some n and $r, r = 1, 2, ..., n$,

$$E(t - X_{r:n} \mid X_{n:n} \le t) \underline{\underline{d}} \; E[X_{n-r+1:n} \mid X_{n:n} \le t], \tag{3.30}$$

if and only if for some $\theta > 0$, $F(x)$ is uniform on $(0, \theta)$.

Proof: See reference.

CHAPTER V

CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS THROUGH DUAL GENERALIZED ORDER STATISTICS

1. INTRODUCTION

Since concept of dual generalized order statistics (*dgos*) was introduced by Pawlas and Szynal (2001), many works on characterization have been carried out by several authors.

Khan *et al.* (2009b) characterized a generalized family of distributions through conditional expectation of *dgos*, when conditioning is on a pair of non-adjacent *dgos*. Further, Khan *et al.* (2010b) established characterizing relationships and used it to characterized a general form of distributions whereas Khan *et al.* (2010c) characterized the generalized family of distribution through conditional variance.

Athar and Faizan (2011) obtained explicit expression for the $r-th$ moment of *dgos* for power function distribution and characterized the same through conditional moments of dual generalized order statistics.

Ahsanullah (2004) and Mbah and Ahsanullah (2007) characterized the uniform and power function distributions based on distributional properties of dual generalized order statistics.

2. CHARACTERIZATION THROUGH MOMENTS

Power function distribution

Theorem 2.1: (Athar and Faizan, 2011)

Let $X'(r,n,m,k)$, $r=1,2,\ldots,n$ be *dgos* based on continuous distribution function $F(\cdot)$. Then for two consecutive values r and $r+1$, $1 \le r < s \le n$, the conditional expectation of *dgos* $X'(s,n,m,k)$ given $X'(r,n,m,k)=x$, is given by

$$E[X'(s,n,m,k) \mid X'(r,n,m,k)=x] = a_{s|r}x, \tag{2.1}$$

if and only if X has the *df*

$$F(x)=\left(\frac{x}{\theta}\right)^{\alpha+1}, \quad 0<x<\theta \tag{2.2}$$

where

$$a_{s|r} = \prod_{i=r+1}^{s} \frac{(\alpha+1)\gamma_i}{1+(\alpha+1)\gamma_i}.$$

Proof: we have

$$g_{s|r} = E[X'(s,n,m,k) \mid X'(r,n,m,k)=x]$$

$$= \frac{C_{s-1}}{C_{r-1}(s-r-1)!}\frac{1}{(m+1)^{s-r-1}}\int_{\alpha}^{\beta} y\left(\frac{F(y)}{F(x)}\right)^{\gamma_s-1}$$

$$\times\left[1-\frac{(F(y))^{m+1}}{(F(x))^{m+1}}\right]^{s-r-1}\frac{f(y)}{F(x)}dy \tag{2.3}$$

Let $u=\dfrac{F(y)}{F(x)}=\left(\dfrac{y}{x}\right)^{\alpha+1}$, then $y = x u^{\frac{1}{(\alpha+1)}}$.

Thus (2.3) becomes

$$E[X'(s,n,m,k) \mid X'(r,n,m,k)=x]$$

$$= \frac{C_{s-1}}{C_{r-1}(s-r-1)!}\frac{1}{(m+1)^{s-r-1}}\int_0^1 x u^{\frac{1}{(\alpha+1)}} u^{\gamma_s-1}[1-u^{m+1}]^{s-r-1}du.$$

Set $u^{m+1}=t$ to get

$$E[X'(s,n,m,k) \mid X'(r,n,m,k)=x]$$

$$= \frac{C_{s-1}}{C_{r-1}(s-r-1)!}\frac{1}{(m+1)^{s-r}}\int_0^1 x t^{\frac{1}{(\alpha+1)(m+1)}+\frac{\gamma_s-1}{m+1}-\frac{m}{m+1}}$$

$$\times[1-t]^{s-r-1}dt$$

$$= a_{s|r}x,$$

where,

$$a_{s|r} = \prod_{i=r+1}^{s} \frac{(\alpha+1)\gamma_i}{1+(\alpha+1)\gamma_i} \text{ (Khan and Alzaid, 2004).}$$

To show that (2.1) implies (2.2), we have

$$g_{s|r+1}(x) - g_{s|r}(x) = a_{s|r+1}x - a_{s|r}x = (a_{s|r+1} - a_{s|r})x$$

$$= \frac{a_{s|r}}{(\alpha+1)\gamma_{r+1}} x.$$

Therefore,

$$\frac{1}{\gamma_{r+1}} \frac{g'_{s|r}(x)}{g_{s|r+1}(x) - g_{s|r}(x)} = \frac{(\alpha+1)}{x} \tag{2.4}$$

and hence

$$\frac{f(x)}{F(x)} = \frac{(\alpha+1)}{x}.$$

implying that

$$F(x) = \left(\frac{x}{\theta}\right)^{\alpha+1}, \quad 0 < x < \theta.$$

General form of distributions

(a) $F(x) = [a\xi(x) + b]^c$

Theorem 2.2: (Khan *et al.*, 2009b)

Let $X'(r,n,\tilde{m},k)$, $i = 1,\ldots,n$ be the *dgos* from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ over the support (α,β), and $h(t)$ be a monotonic and differentiable function of t. If for two consecutive values r and $r+1$, $2 \le r+1 < j < s \le n$

$g_{j|l,s}(x,y)$

$$= E[h(X'(j,n,\tilde{m},k)) \mid X'(l,n,\tilde{m},k) = x, X'(s,n,\tilde{m},k) = y], \quad l = r, r+1 \tag{2.5}$$

exist and $g(x,y)$ is a finite and differentiable function of x, then

$$\gamma_{r+1}\frac{f(x)}{F(x)}+\frac{\frac{\partial}{\partial x}B_r^s(x,y)}{B_r^s(x,y)}=\frac{\frac{\partial}{\partial x}g_{j|r,s}(x,y)}{[g_{j|r+1,s}(x,y)-g_{j|r,s}(x,y)]} \tag{2.6}$$

and

$$\frac{[F(x)]^{\gamma_{r+1}}B_r^s(x,y)}{B_r^s(x,y)}\exp\left[-\int_x^{\beta}A_1(t,y)dt\right], \tag{2.7}$$

where

$$B_r^s(x,y)=\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left\{\frac{F(y)}{F(x)}\right\}^{\gamma_i}\right], \tag{2.8}$$

and

$$A_1(x,y)=\frac{\frac{\partial}{\partial x}g_{j|r,s}(x,y)}{[g_{j|r+1,s}(x,y)-g_{j|r,s}(x,y)]} \tag{2.9}$$

Proof: we have, in view of (5.6) and (2.5)

$$g_{j|r,s}(x,y)B_r^s(x,y)=\int_y^x h(t)B_r^j(x,t)B_r^s(t,y)\frac{f(t)}{F(t)}dt \tag{2.10}$$

Differentiate both the sides *w.r.t.* x, to get

$$\frac{\partial}{\partial x}g_{j|r,s}(x,y)B_r^s(x,y)+g_{j|r,s}(x,y)\left[\frac{\partial}{\partial x}B_r^s(x,y)\right]$$

$$=\int_y^x h(t)\left[\frac{\partial}{\partial x}B_r^j(x,t)\right][B_j^s(t,y)]\frac{f(t)}{F(t)}dt \tag{2.11}$$

after noting that $B_r^s(x,x)=0$, as $\sum\limits_{i=r+1}^{s}a_i^{(r)}(s)=0$ (Khan *et al.*, 2006).

Since $a_i^{(r+1)}(s)=(\gamma_{r+1}-\gamma_i)a_i^{(r)}(s),\ \ i=r+2,\ldots,s.$

Hence,

$$B_{r+1}^{s}(x,y)=\left[\sum_{i=r+2}^{s} a_i^{(r+1)}(s)\left\{\frac{F(y)}{F(x)}\right\}^{\gamma_i}\right]$$

$$=\gamma_{r+1}B_r^s(x,y)-\frac{F(x)}{f(x)}\left[\frac{\partial}{\partial x}B_r^s(x,y)\right] \tag{2.12}$$

Thus (2.11) reduces to

$$\frac{\partial}{\partial x}g_{j|r,s}(x,y)B_r^s(x,y)$$

$$=[g_{j|r+1,s}(x,y)-g_{j|r,s}(x,y)]B_{r+1}^s(x,y)\frac{f(x)}{F(x)}$$

or,

$$\frac{f(x)}{F(x)}\frac{B_{r+1}^s(x,y)}{B_r^s(x,y)}=\frac{\frac{\partial}{\partial x}g_{j|r,s}(x,y)}{[g_{j|r+1,s}(x,y)-g_{j|r,s}(x,y)]}=A_1(x,y) \tag{2.13}$$

implying that

$$\gamma_{r+1}\frac{f(x)}{F(x)}+\frac{\frac{\partial}{\partial x}B_r^s(x,y)}{B_r^s(x,y)}=\frac{\frac{\partial}{\partial x}g_{j|r,s}(x,y)}{[g_{j|r+1,s}(x,y)-g_{j|r,s}(x,y)]} \tag{2.14}$$

Integrating both the sides *w.r.t.* x over (x,β), we get (2.7).

Corollary 2.1: (Khan *et al.*, 2009b)

It may be noted that for $\gamma_i \neq \gamma_j$ and $m_1=\cdots=m_{n-1}=m\neq -1$.

$$a_i^{(r)}(s)=\frac{1}{\prod\limits_{\substack{j=r+1\\ i\neq j}}^{s}(\gamma_j-\gamma_i)}$$

$$=(-1)^{s-i}\frac{1}{(m+1)^{s-r-1}}\frac{1}{(i-r-1)!(s-i)!}. \tag{2.15}$$

Thus,

$$\frac{[F(x)]^{\gamma_{r+1}} B_r^s(x,y)}{B_r^s(\beta,y)}$$

$$= \frac{[F(x)]^{\gamma_{r+1}} \left(\frac{F(y)}{F(x)}\right)^{\gamma_s} \frac{1}{(m+1)^{s-r-1}(s-r-1)!}}{[F(x)]^{\gamma_s} \frac{1}{(m+1)^{s-r-1}(s-r-1)!}}$$

$$\times \frac{\sum_{i=r+1}^{s}(-1)^{s-i} \frac{(s-r-1)!}{(i-r-1)!(s-i)!} \left(\frac{F(y)}{F(x)}\right)^{\gamma_i-\gamma_s}}{\sum_{i=r+1}^{s}(-1)^{s-i} \frac{(s-r-1)!}{(i-r-1)!(s-i)!} [F(y)]^{\gamma_i-\gamma_s}}$$

$$= \frac{[F(x)^{m+1}]^{(s-r-1)} \left[1-\frac{F(y)^{m+1}}{F(x)^{m+1}}\right]^{(s-r-1)}}{[1-F(y)^{m+1}]^{(s-r-1)}}$$

implying that

$$\frac{1-[F(x)]^{m+1}}{1-[F(y)]^{m+1}} = 1-\exp\left[-\frac{1}{(s-r-1)}\int_x^\beta A_1(t,y)dt\right], \quad m>-1 \tag{2.16}$$

at

$$\frac{\log F(x)}{\log F(y)} = 1-\exp\left[-\frac{1}{(s-r-1)}\int_x^\beta A_1(t,y)dt\right], \quad m=-1 \tag{2.17}$$

as

$$\frac{\partial}{\partial m}\{F(x)\}^{m+1} = \{F(x)\}^{m+1}\log F(x),$$

which tends to $\log F(x)$ as $m \to -1$.

Theorem 2.3: (Khan *et al.*, 2009b)

Let $X'(i,n,\tilde{m},k)$, $i=1,\ldots,n$ be the *dgos* from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ over the support (α,β) and $h(t)$ be a monotonic and differentiable function of t. If for two consecutive values $s-1$ and s, $1 \le r < j < s-1 < n$,

$$g_{j|r,l}(x,y)$$
$$= E[h(X'(j,n,\tilde{m},k)) \mid X'(r,n,\tilde{m},k)=x, X'(s,n,\tilde{m},k)=y], \quad l=s-1,s$$

exist, then

$$\gamma_s \frac{f(y)}{F(y)} - \frac{\frac{\partial}{\partial y} B_r^s(x,y)}{B_r^s(x,y)} = \frac{\frac{\partial}{\partial y} g_{j|r,s}(x,y)}{[g_{j|r,s}(x,y) - g_{j|r,s-1}(x,y)]} \tag{2.18}$$

and

$$\sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i - \gamma_s} = a_s^{(r)}(s) \exp\left[- \int_\alpha^y A_2(x,t)dt \right], \tag{2.19}$$

where $B_r^s(x,y)$ is as defined in (2.8),

$$A_2(x,y) = \frac{\frac{\partial}{\partial y} g_{j|r,s}(x,y)}{[g_{j|r,s}(x,y) - g_{j|r,s-1}(x,y)]}. \tag{2.20}$$

Proof: Differentiating both sides of (2.10) *w.r.t.* y and proceeding as in the Theorem 2.2, we get

$$\frac{f(y)}{F(y)} - \frac{B_r^{s-1}(x,y)}{B_r^s(x,y)} = \frac{\frac{\partial}{\partial y} g_{j|r,s}(x,y)}{[g_{j|r,s}(x,y) - g_{j|r,s-1}(x,y)]} \tag{2.21}$$

where

$$B_r^{s-1}(x,y) = \left(\gamma_s B_r^s(x,y) - \frac{F(y)}{f(y)} \frac{\partial}{\partial y} B_r^s(x,y) \right)$$

as

$$a_i^{(r)}(s-1) = (\gamma_s - \gamma_i)a_i^{(r)}(s)$$

That is,

$$\gamma_s \frac{f(y)}{F(y)} - \frac{\frac{\partial}{\partial y} B_r^s(x,y)}{B_r^s(x,y)} = \frac{\frac{\partial}{\partial y} g_{j|r,s}(x,y)}{[g_{j|r,s}(x,y) - g_{j|r,s-1}(x,y)]}$$

Therefore,

$$\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{\gamma_i - \gamma_s} = a_s^{(r)}(s)\exp\left[-\int_\alpha^y A_2(x,t)dt\right],$$

and hence the result.

Corollary 2.2: (Khan *et al.*, 2009b)

It may be noted that at $\gamma_i \neq \gamma_j$ but $m_1 = \cdots = m_{n-1} = m > -1$.

$$\frac{\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{\gamma_i - \gamma_s}}{a_s^{(r)}(s)} = \left[1 - \frac{\{F(y)\}^{m+1}}{\{F(x)\}^{m+1}}\right]^{s-r-1}$$

implying that

$$\frac{\{F(y)\}^{m+1}}{\{F(x)\}^{m+1}} = 1 - \exp\left(-\frac{1}{s-r-1}\int_\alpha^y A_2(x,t)dt\right), \; m > -1 \tag{2.22}$$

Remark 2.1: In the limiting case as $x \to \beta$, at $r = 0$, Theorem 2.3 reduces to

$$\sum_{i=1}^{s} a_i(s)[F(y)]^{\gamma_i - \gamma_s} = a_s(s)\exp\left[-\int_\alpha^y \frac{g'_{j|t}(t)}{[g_{j|s}(t) - g_{j|s-1}(t)]}dt\right],$$

where $g_{j|s}(y) = E[h(X(j,n,\tilde{m},k)) \mid X(s,n,\tilde{m},k) = y]$, for $\gamma_i \neq \gamma_j$ and for $m_1 = \cdots = m_{n-1} = m > -1$

$$[F(y)]^{m+1} = 1 - \exp\left[-\frac{1}{s-1}\int_\alpha^y \frac{g'_{j|s}(t)}{[g_{j|s}(t) - g_{j|s-1}(t)]}dt\right]$$

as given by Khan *et al.* (2010b).

Examples:

For adjacent *dgos* at $j = r+1,\ s = r+2$ and $m_{r+1} > -1$, it can be seen that

$$f_{r+1|r,r+2}(t \mid x, y) = \frac{(m_{r+1}+1)\{F(t)\}^{m_{r+1}} f(t)}{[\{F(x)\}^{m_{r+1}+1} - \{F(y)\}^{m_{r+1}+1}]}, \tag{2.23}$$

and therefore, corresponding to (2.16) and (2.22), we have respectively

$$\frac{1-\{F(x)\}^{m_{r+1}+1}}{1-\{F(y)\}^{m_{r+1}+1}} = 1 - e^{-I_1} \tag{2.24}$$

and

$$\frac{\{F(y)\}^{m_{r+1}+1}}{\{F(x)\}^{m_{r+1}+1}} = 1 - e^{-I_2}, \tag{2.25}$$

where

$$I_1 = \int_x^{\beta} A_1(t, y)dt, \quad I_2 = \int_{\alpha}^{y} A_2(x, t)dt.$$

Thus we have,

$$g_{r|r,s}(x, y)$$

$$= E[h(X'(r, n, \tilde{m}, k)) \mid X'(r, n, \tilde{m}, k) = x, X'(s, n, \tilde{m}, k) = y] = h(x)$$

$$g_{s|r,s}(x, y)$$

$$= E[h(X'(s, n, \tilde{m}, k)) \mid X'(r, n, \tilde{m}, k) = x, X'(s, n, \tilde{m}, k) = y] = h(y)$$

and

$$g_{r+1|r,r+2}(x, y)$$

$$= E[h(X'(r+1, n, \tilde{m}, k)) \mid X'(r, n, \tilde{m}, k) = x, X'(r+2, n, \tilde{m}, k) = y]$$

$$= g(x, y). \tag{2.26}$$

Therefore,

$$A_1(x, y) = \frac{\frac{\partial}{\partial x} g(x, y)}{[h(x) - g(x, y)]} \tag{2.27}$$

and

$$A_2(x,y) = \frac{\frac{\partial}{\partial y} g(x,y)}{[g(x,y) - h(y)]} \tag{2.28}$$

(i) $$g(x,y) = \frac{c}{a(c+1)} \frac{[ah(y)+b]^{c+1} - [ah(x)+b]^{c+1}}{[ah(y)+b]^c - [ah(x)+b]^c} - \frac{b}{a}, \; c \neq -1 \tag{2.29}$$

if and only if

$$1 - \{F(x)\}^{m_{r+1}+1} = [ah(x)+b]^c, \tag{2.30}$$

(ii) $$g(x,y) = \frac{c}{(c-1)} \frac{[\{h(x)\}^{c-1} - \{h(y)\}^{c-1}]}{[h(x)]^c - [h(y)]^c}, \; c \neq 1 \tag{2.31}$$

if and only if

$$1 - \{F(x)\}^{m_{r+1}+1} = a[h(x)]^{-c} + b, \tag{2.32}$$

(iii) $$g(x,y) = \frac{c}{a(c+1)} \frac{[ah(x)+b]^{c+1} - [ah(y)+b]^{c+1}}{[ah(x)+b]^c - [ah(y)+b]^c} - \frac{b}{a}, \; c \neq -1 \tag{2.33}$$

if and only if

$$\{F(x)\}^{m_{r+1}+1} = [ah(x)+b]^c, \tag{2.34}$$

(iv) For $m_1 = \cdots = m_{n-1} = m > -1$

$$g_{j|r,s}(x,y) = \frac{(s-j)h(x) + (j-r)h(y)}{(s-r)} \tag{2.35}$$

if and only if

$$1 - \{F(x)\}^{m+1} = ah(x) + b, \tag{2.36}$$

where a, b, c and $h(x)$ are so chosen that $F(x)$ is a *df.*

Proof: See reference.

Theorem 2.4: (Khan *et al.*, 2010b)

Let $\xi(x)$ be a monotonic and continuous function of x. If for $1 \le r < s-1 < n$

$$E[\xi\{X'(s,n,m,k)\} \mid X'(l,n,m,k) = x] = g_{s|l}(x), \quad l = r, r+1, \tag{2.37}$$

exist and is differentiable with respect to x, then

$$F(x) = \exp\left[-\frac{1}{\gamma_{r+1}} \int_x^\beta \frac{g'_{s|r}(t)}{g_{s|r+1}(t) - g_{s|r}(t)} dt \right], \quad \alpha < x < \beta. \tag{2.38}$$

Proof: We have,

$$E[\xi\{X'(s,n,m,k)\} \mid X'(r,n,m,k) = x] = g_{s|r}(x)$$

That is,

$$\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}} \int_\alpha^x \xi(y)[(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-1}$$

$$\times [F(y)]^{\gamma_s - 1} f(y) dy$$

$$= g_{s|r}(x)[F(x)]^{\gamma_{r+1}} \tag{2.39}$$

Differentiating (2.5) both the sides with respect to x, we get

$$[F(x)]^{\gamma_{r+2}} [F(x)]^m \gamma_{r+1} f(x) g_{s|r+1}(x)$$

$$= g'_{s|r}(x)[F(x)]^{\gamma_{r+1}} + \gamma_{r+1} g_{s|r}(x)[F(x)]^{\gamma_{r+1}-1} f(x)$$

$$\frac{f(x)}{F(x)} = \frac{1}{\gamma_{r+1}} \left[\frac{g'_{s|r}(x)}{g_{s|r+1}(x) - g_{s|r}(x)} \right]$$

and hence the Theorem.

Corollary 2.3: (Khan *et al.*, 2010b)

$$E[X'(s,n,m,k) \mid X'(r,n,m,k) = x] = a^*_{s|r} x + b^*_{s|r} = g_{s|r}(x) \tag{2.40}$$

if and only if

$$F(x) = [ax+b]^c, \ \alpha < x < \beta \tag{2.41}$$

where

$$a^*_{s|r} = \prod_{i=r+1}^{s} \frac{c\gamma_i}{1+c\gamma_i} \text{ and } b^*_{s|r} = -\frac{b}{a}(1-a^*_{s|r}) \tag{2.42}$$

where a, b and c are such that $F(x)$ is a *df* over (α, β).

Proof: We have,

$$E[X'(s,n,m,k) \mid X'(r,n,m,k) = x]$$

$$= \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}} \int_{\alpha}^{x} y \left[\frac{F(y)}{F(x)}\right]^{\gamma_s - 1} \left[1 - \frac{(F(y))^{m+1}}{(F(x))^{m+1}}\right]^{s-r-1} \times \frac{f(y)}{F(x)} dy \tag{2.43}$$

Let

$$\left[\frac{F(y)}{F(x)}\right]^{m+1} = \left[\frac{ay+b}{ax+b}\right]^{(m+1)c} = t$$

then (2.43) reduces to

$$E[X'(s,n,m,k) \mid X'(r,n,m,k) = x]$$

$$= \frac{C_{s-1}}{C_{r-1}(s-r-1)!} \frac{1}{(m+1)^{s-r-1}} \int_0^1 \frac{(ax+b)t^{\frac{1}{c(m+1)}} - b}{a} t^{\frac{\gamma_s - 1}{m+1} - \frac{m}{m+1}} \times [1-t]^{s-r-1} dt$$

$$= a^*_{s|r} x - \frac{b}{a}(1 - a^*_{s|r} x)$$

where

$$a^*_{s|r} = \prod_{i=r+1}^{s} \frac{c\gamma_i}{1+c\gamma_i} \text{ (Khan and Alzaid, 2004)}$$

To show that (2.40) implies (2.41), we have

$$g_{s|r+1}(x) - g_{s|r}(x) = (a^*_{s|r+1} - a^*_{s|r})\left(x + \frac{b}{a}\right)$$

$$= \frac{a^*_{s|r}}{ac\gamma_{r+1}}(ax+b) \text{ as } a^*_{s|r+1} = \prod_{i=r+2}^{s} \frac{c\gamma_i}{1+c\gamma_i} = \frac{1+c\gamma_{r+1}}{c\gamma_{r+1}} a^*_{s|r}$$

Therefore,

$$\frac{1}{\gamma_{r+1}}\left[\frac{g'_{s|r}(x)}{g_{s|r+1}(x) - g_{s|r}(x)}\right] = \frac{ac}{ax+b}$$

and hence

$$\frac{f(x)}{F(x)} = \frac{ac}{ax+b}$$

implying that

$$F(x) = [ax+b]^c.$$

Remark 2.2: Let $\xi(x)$ be a monotonic and continuous function of x, then

$$E[\xi\{X'(s,n,m,k)\} \mid X'(r,n,m,k) = x] = a^*_{s|r}\xi(x) + b^*_{s|r} \tag{2.44}$$

if and only if

$$F(x) = [a\xi(x)+b]^c, \ \alpha < x < \beta \tag{2.45}$$

where a, b and c are such that $F(x)$ is a *df* over (α, β).

This can be proved on the lines of the Corollary 2.3 by considering

$$g_{s|r}(x) = a^*_{s|r}\xi(x) + b^*_{s|r}.$$

A number of distributions can be characterized by the proper choice of a, b, c and $\xi(x)$.

Remark 2.3: If we denote $X'_{1:n} \geq X'_{2:n} \geq \cdots \geq X'_{n:n}$ (lower order statistics) and $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ (order statistics), then for

$$F(x) = [a\xi(x) + b]^c$$

$$E[\xi(X'_{s:n}) \mid X'_{r:n} = x] = a^*_{s|r}\xi(x) + b^*_{s|r} = E[\xi(X_{n-s+1:n}) \mid X_{n-r+1:n} = x]$$

(Burkschat *et al.*, 2003)

where $a^*_{s|r} = \prod_{j=r+1}^{s} \frac{c(n-j+1)}{c(n-j+1)+1}$ and $b^*_{s|r} = -\frac{b}{a}(1 - a^*_{s|r})$.

Replacing $n-s+1$ by r, we get

$$E[\xi(X_{r:n}) \mid X_{s;n} = y] = a^*_2\xi(y) + b^*_2$$

if and only if

$$F(y) = [a\xi(y) + b]^c$$

where $a^*_2 = \prod_{j=n-s+2}^{n-r+1} \frac{c(n-j+1)}{c(n-j+1)+1} = \prod_{l=1}^{s-r} \frac{c(s-l)}{c(s-l)+1}$ and $b^*_2 = -\frac{b}{a}(1 - a^*_2)$

as obtained by Khan and Abouammoh (2000).

And for the lower k – record values $(m = -1)$

$$E[\xi\{X'(s,n,-1,k)\} \mid X'(r,n,-1,k) = x] = a^*_{s|r}\xi(x) + b^*_{s|r}$$

where $a^*_{s|r} = \left(\frac{ck}{1+ck}\right)^{s-r}$ and $b^*_{s|r} = -\frac{b}{a}(1 - a^*_{s|r})$.

Theorem 2.5: (Khan *et al.*, 2010b)

Let $\xi(x)$ be a monotonic and continuous function of x. If for $1 \leq r < s-1 < n$

$$E[\xi\{X'(r,n,m,k)\} \mid X'(l,n,m,k) = y] = g_{r|l}(y),\ l = s-1, s \tag{2.46}$$

exist and is differentiable with respect to y, then

$$\frac{(m+1)f(y)[F(y)]^m}{1-[F(y)]^{m+1}} = \frac{g'_{s|r}(y)}{(s-1)[g_{r|s}(y)-g_{r|s-1}(y)]} = A(y),$$

$$m \neq -1 \tag{2.47}$$

$$-\frac{f(y)}{F(y)\log F(y)} = A(y), \ m=-1 \tag{2.48}$$

and

$$F(y) = \left[1-\exp\left(-\int_{\alpha}^{y} A(t)dt\right)\right]^{\frac{1}{m+1}}, \ \alpha < x < \beta, \ m \neq -1 \tag{2.49}$$

$$-\log F(y) = e^{-\int_{p}^{y} A(t)dt}, \ m=-1 \tag{2.50}$$

where

$$-\log F(p) = 1.$$

Proof: See reference.

Corollary 2.4: (Khan *et al.*, 2010b)

$$E[X'(r,n,m,k) \mid X'(s,n,m,k)=y] = a^*_{r|s}y + b^*_{r|s} = g_{r|s}(y) \tag{2.51}$$

if and only if

$$1-[F(x)]^{m+1} = [ay+b]^c, \ \alpha < y < \beta, \ m \neq -1 \tag{2.52}$$

where $F(y)$ is a *df* over (α, β),

$$a^*_{r|s} = \prod_{j=1}^{s-r} \frac{c(s-j)}{1+c(s-j)}, \text{ and } b^*_{r|s} = -\frac{b}{a}(1-a^*_{r|s})$$

and

$$-\log F(y) = [ay+b]^c \text{ at } m=-1 \text{ with } -\log F(p) = 1.$$

Proof: See reference.

Remark 2.4: As noted in the Remark 2.3, for

$$F(y) = [a\xi(y) + b]^c$$

$$E[\xi(X'_{r:n}) \mid X'_{s:n} = y] = a^*_{r|s}\xi(y) + b^*_{r|s} = E[\xi(X_{n-r+1:n}) \mid X_{n-s+1:n} = y]$$

where $a^*_{r|s} = \prod_{j=1}^{s-r} \frac{c(s-j)}{c(s-j)+1}$ and $b^*_{r|s} = -\frac{b}{a}(1 - a^*_{r|s})$.

Replacing $n-r+1$ by s, we get

$$E[\xi(X_{s:n}) \mid X_{r:n} = x] = a^*_1\xi(x) + b^*_1$$

if and only if

$$F(x) = [a\xi(x) + b]^c$$

where $a^*_1 = \prod_{j=1}^{s-r} \frac{c(n-r+1-j)}{c(n-r+1-j)+1} = \prod_{l=r+1}^{s} \frac{c(n-l+1)}{c(n-l+1)+1}$ and $b^*_1 = -\frac{b}{a}(1 - a^*_1)$

as obtained by Khan and Abouammoh (2000).

Theorem 2.6: (Khan *et al.*, 2010b)

Let $\xi(x)$ be a monotonic and continuous function of x. If for $1 \le r < s-1 < n$

$$E[\xi\{X'(s,n,\tilde{m},k)\} \mid X'(l,n,\tilde{m},k) = x] = g_{s|l}(x), \quad l = r, r+1, \tag{2.53}$$

exist and is differentiable with respect to x, then

$$F(x) = e^{-\int_x^\beta A(t)dt}$$

where

$$A(t) = \frac{g'_{s|r}(t)}{\gamma_{r+1}[g_{s|r+1}(t) - g_{s|r}(t)]}.$$

Proof: It can be seen that

$$g_{s|r+1}(x) = g_{s|r}(x) - \frac{C_{s-1}}{C_r} \sum_{i=r+1}^{s} \gamma_i a_i^{(r)}(s) \int_\alpha^x \frac{\xi(y)[F(y)]^{\gamma_i - 1} f(y)}{[F(x)]^{\gamma_i}} dy \tag{2.54}$$

Now,

$$g_{s|r}(x)=\frac{C_{s-1}}{C_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_{\alpha}^{x}\frac{\xi(y)[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}dy \tag{2.55}$$

Differentiate both the sides with respect to x to get,

$$g'_{s|r}(x)=\frac{C_{s-r}}{C_{r-1}}\frac{\xi(x)f(x)}{F(x)}\sum_{i=r+1}^{s}a_i^{(r)}(s)-\frac{f(x)}{F(x)}\frac{\gamma_{r+1}C_{s-r}}{C_r}\sum_{i=r+1}^{s}\gamma_i a_i^{(r)}(s)$$

$$\times\int_{\alpha}^{x}\frac{\xi(y)[F(y)]^{\gamma_i-1}}{[F(x)]^{\gamma_i}}f(y)dy$$

In view of the results $\sum_{i=r+1}^{s}a_i^{(r)}(s)=0,\ 1\le r<s$ (Khan and Alzaid, 2004) and (2.54), we have

$$g_{s|r+1}=g_{s|r}+\frac{1}{\gamma_{r+1}}\frac{F(x)}{f(x)}g'_{s|r}$$

Therefore,

$$\frac{f(x)}{\overline{F}(x)}=\frac{g'_{s|r}(x)}{\gamma_{r+1}[g_{s|r+1}(x)-g_{s|r}(x)]}=A(x)$$

and hence the Theorem.

Comparing the Theorem 2.4 and 2.6, we notice that both the results are identical. Thus proceeding as in the Corollary 2.3 and the Remark 2.2, it can be seen that

$$F(x)=[ax+b]^c \text{ if and only if } g_{s|r}(x)=a^*_{s|r}x+b^*_{s|r} \tag{2.56}$$

and

$$F(x)=[a\xi(x)+b]^c \text{ if and only if } g_{s|r}(x)=a^*_{s|r}x+b^*_{s|r} \tag{2.57}$$

where $a^*_{s|r}$ and $b^*_{s|r}$ are as given in the Corollary 2.3.

Theorem 2.7: (Khan *et al.*, 2010b)

Let $\xi(x)$ be a monotonic and continuous function of x. If for $1 \le r < s-1 < n$

$$g_{r|l}(y) = E[\xi\{X'(r,n,\tilde{m},k)\} \mid X'(l,n,\tilde{m},k) = y], \; l = s, s-1 \qquad (2.58)$$

exist and is differentiable with respect to y, then

$$\frac{\gamma_s f(y)}{F(y)} - \frac{B_s'(y)}{B_s(y)} = \frac{g_{r|s}'(y)}{[g_{r|s}(y) - g_{r|s-1}(y)]} = D(y)$$

and

$$\sum_{i=1}^{s} a_i(s)[F(y)]^{\gamma_i - \gamma_s} = a_s(s) e^{-\int_\alpha^y D(t)dt}, \; \alpha < y < \beta$$

where $\sum_{i=1}^{s} a_i(s) = 0$

$$B_s(y) = \sum_{i=1}^{s} a_i(s)[F(y)]^{\gamma_i}$$

$$a_s(s) = \prod_{\substack{j=1 \\ j \ne s}}^{s} \frac{1}{(\gamma_j - \gamma_s)} = \prod_{j=1}^{s-1} \frac{1}{(\gamma_j - \gamma_s)} \qquad (2.59)$$

$F(y)$ is a *df* over (α, β), $D(y)$ and $B_s(y)$ are finite and differentiable functions of y.

Proof: See reference.

(b) $\quad F(x) = e^{-ah(x)}, \; x \in (\alpha, \beta)$

Lemma 2.1: (Khan *et al.*, 2010c)

Let the *df* $F(x)$ be twice differentiable on (α, β), and let $h(x)$ be a non-increasing and a twice differentiable function of x such that $h(x) \to 0$ as $x \to \beta$. Then the solution of the differential equation

$$\frac{F''(x)}{F(x)}+(\gamma_{r+1}-1)\left[\frac{F'(x)}{F(x)}\right]^2-\frac{h''(x)}{h'(x)}\left[\frac{F'(x)}{F(x)}\right]-\alpha^2\gamma_{r+1}[h'(x)]^2=0 \quad (2.60)$$

is

$$F(x)=e^{-ah(x)}, \text{ for all } x\in(\alpha,\beta), \text{ where } a\neq 0 \text{ is a constant.} \quad (2.61)$$

Proof: See reference.

Theorem 2.8: (Khan *et al.*, 2010c)

Let X be a continuous random variable with *df* $F(x)$ and *pdf* $f(x)$ over the support (α,β). Let $E[h(x)]^2$ exists, then for some $0<r<n$,

$$V[h\{X'(r+1,n,m,k)\}\,|\,X'(r,n,m,k)=x]=\frac{1}{a^2\gamma_{r+1}^2}$$

if and only if

$$F(x)=e^{-ah(x)}, \quad (2.62)$$

where $a\neq 0$, $h(y)$ be a non-increasing and a twice differentiable function of y such that $h(y)\to 0$ as $y\to\beta$ and $h(y)F(y)\to 0$ as $y\to\alpha$.

Proof: See reference.

Remark 2.5: At $m=0, k=1$ and $\gamma_r=n-r+1$, Theorem 2.8 reduces for order statistics.

3. CHARACTERIZATION THROUGH DISTRIBUTIONAL PROPERTY

Uniform distribution

Lemma 3.1: (Ahsanullah , 2004)

$$F_{l,r,n,m,k}(x)=I_{\alpha(x)}\left(r,\frac{\gamma_r}{m+1}\right) \quad \text{if } m>-1 \quad (3.1)$$

and

$$F_{l,r,n,m,k}(x)=\Gamma_{\beta(x)}(r) \quad \text{if } m=-1 \quad (3.2)$$

where, $F_{l,r,n,m,k}(x) = F_{X'(r,n,m,k)}$

$$\alpha(x) = 1 - [F(x)]^{m+1},$$

$$\beta(x) = -\ln F(x),$$

$$I_x(p,q) = \frac{1}{B(p,q)} \int_0^x u^{p-1}(1-u)^{q-1}du, \; B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

$$\Gamma_x(r) = \int_0^\infty \frac{1}{\Gamma(r)} u^{r-1}e^{-u}du.$$

Proof: For $m > -1$

$$F_{l,r,n,m,k}(x) = \frac{C_{r-1}}{\Gamma(r)} \int_0^x [F(u)]^{\gamma_r - 1}[g_m(F(u)]^{r-1} f(u)du$$

$$= \frac{C_r}{\Gamma(r)} \int_0^x [F(u)]^{\gamma_r - 1}\left[\frac{1-[F(u)]^{m+1}}{m+1}\right]^{r-1} f(u)du$$

$$= \frac{1}{\Gamma(r)} \int_0^{1-[F(x)]^{m+1}} (1-u)^{\frac{\gamma_r}{m+1}} - 1(-u)^{r-1}du$$

$$= I_{\alpha(x)}\left(\frac{r, \gamma_r}{m+1}\right)$$

$$\alpha(x) = 1 - [F(x)]^{m+1}.$$

For $m = -1, \gamma_j = k, \;\; j = 0,1,\cdots,n$

$$F_{l,r,n,m,k}(x) = \frac{k^r}{\Gamma(r)} \int_{-\infty}^x [F(u)]^{k-1}[-\ln F(u)]^{r-1} f(u)du$$

$$= \int_0^{-k\ln F(x)} \frac{1}{\Gamma(r)} u^{r-1}e^{-u}du$$

$$\Gamma_{\beta(n)}(r), \beta(x) = -k\ln F(x).$$

Lemma 3.2: (Ahsanullah , 2004)

For $m > -1$

$$\gamma_{r+1}[F_{l,r,n,m,k}(x) - F_{l,r+1,n,m,k}(x)] = \frac{F(x)}{f(x)} f_{l,r+1,n,m,k}(x)$$

and for $m=-1$

$$\gamma_{r+1}[F_{l,r,n,m,k}(x)-F_{l,r+1,n,m,k}(x)]=\frac{F(x)}{f(x)}f_{l,r+1,n,m,k}(x)$$

Proof: For $m>-1$

$$F_{l,r,n,m,k}(x)-F_{l,r+1,n,m,k}(x)$$

$$=I_{\alpha(x)}\left(r,\frac{\gamma_r}{m+1}\right)-I_{\alpha(x)}\left(r+1,\frac{\gamma_{r+1}}{m+1}\right)$$

$$=I_{\alpha(x)}\left(r,\frac{\gamma_r}{m+1}\right)-I_{\alpha(x)}\left(r+1,\frac{\gamma_r}{m+1}-1\right).$$

We know that

$$I_x(a,b)-I_x(a+1,b-1)=\frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)}x^a(1-x)^{b-1},$$

thus,

$$F_{l,r,n,m,k}(x)-F_{l,r+1,n,m,k}(x)=\frac{\Gamma\left(r,\frac{\gamma_r}{m+1}\right)}{\Gamma(r+1)\Gamma\left(\frac{\gamma_r}{m+1}\right)}$$

$$\times[1-(F(x))^{m+1}]^r[F(x)^{m+1}]^{\frac{\gamma_r}{m+1}-1}$$

$$=\frac{\gamma_{1,\ldots,}\gamma_r}{\Gamma(r+1)}\left(\frac{1-(F(x))^{m+1}}{m+1}\right)^r[F(x)]^{\gamma_{r+1}}$$

$$=\frac{F(x)}{\gamma_{r+1}f(x)}f_{l,r+1,n,m,k}(x).$$

Therefore ,

$$\gamma_{r+1}[F_{l,r,n,m,k}(x)-F_{l,r+1,n,m,k}(x)]=\frac{F(x)}{f(x)}f_{l,r+1,n,m,k}(x).$$

For $m=-1$, the proof of the result

$$k[F_{l,r,n,m,k}(x) - F_{l,r+1,n,m,k}(x)] = \frac{F(x)}{f(x)} f_{l,r+1,n,m,k}(x)$$

is similar.

Theorem 3.1: (Ahsanullah, 2004)

Let X be an absolutely continuous (with respect to Lebesque Measure) bounded random variable with strictly increasing *cdf* $F(x)$ and *pdf* $f(x)$. Without any loss of generality we will take $F(0)=0$ and $F(1)=1$. Then the following two statements are equivalent

(a) X is uniformly distributed random variable in $(0,1)$

(b) $X'(r+1,n,m,k) \underline{\underline{d}} X'(r,n,m,k)\, W_{r+1}$,

where W_{r+1} is independent of $X'(r+1,n,m,k)$ and the *pdf* of W_{r+1} is

$$f_{r+1}(w) = \lambda_{r+1} w^{\lambda_{r+1}-1},\ 0 < w < 1.$$

Proof: The proof of (a) implies (b) can be obtained from the dual representation of dual generalized order statistics in Burkschat *et al.* (2003). However, here we present a direct proof of result.

Let $m > -1$ and $Y = X'(r,n,m,k)\, W_{r+1}$

$$F_Y(x) = P(Y \le x) = P\{X_l(r,n,m,k)\, W_{r+1} \le x\}$$

$$F_Y(x) = \int_0^x f_{l,r,n,m,k}(u)\, du + \int_x^1 \left(\frac{x}{u}\right)^{\lambda_{r+1}} f_{l,r,n,m,k}(u)\, du$$

$$= F_{l,r,n,m,k}(x) + \int_x^1 \left(\frac{x}{u}\right)^{\lambda_{r+1}} f_{l,r,n,m,k}(u)\, du. \tag{3.3}$$

Differentiating (2.3) *w.r.t.* x, we obtain,

$$f_Y(x) = f_{l,r,n,m,k}(x) - f_{l,r,n,m,k}(x)u$$

$$+ \int_x^1 \frac{\gamma_{r+1}}{u^{\gamma_{r+1}}}(x)^{\lambda_{r+1}-1} f_{l,r,n,m,k}(u)du \tag{3.4}$$

From (3.4), we obtain on simplification

$$\frac{f_Y(x)}{(x)^{\lambda_{r+1}-1}} = \int_x^1 \frac{\gamma_{r+1}}{u^{\gamma_{r+1}}} f_{l,r,n,m,k}(u)\, du. \tag{3.5}$$

Differentiating both sides of (2.5) *w.r.t.* x, we obtain,

$$\frac{f'_Y(x)}{(x)^{\lambda_{r+1}-1}} - \frac{f_Y(x)}{(x)^{\lambda_{r+1}}}(\gamma_{r+1}-1) = -\frac{\gamma_{r+1}\ f_{l,r,n,m,k}(x)}{x^{\gamma_{r+1}}}. \tag{3.6}$$

On simplification from (3.6), we get

$$f''_Y(x)' - \frac{\gamma_{r+1}-1}{x} f_Y(x) = -\frac{c_r\ x^{\gamma_{r-1}}}{\Gamma(r)x}\left[\frac{1-x^{m+1}}{m+1}\right]^{r-1}. \tag{3.7}$$

Multiplying both sides of (3.7) by $x^{-(\gamma_{r+1}-1)}$, we obtain

$$\frac{d}{dx}[f_Y(x)x^{-(\gamma_{r+1}-1)}] = -\frac{c_r\ x^m}{\Gamma(r)}\left[\frac{1-x^{m+1}}{m+1}\right]^{r-1}.$$

Thus,

$$f_Y(x)x^{-(\gamma_{r+1}-1)}) = c - \int \frac{c_r\ x^m}{\Gamma(r)}\left[\frac{1-x^{m+1}}{m+1}\right]^{r-1} dx$$

$$= c + \frac{c_r}{\Gamma(r+1)}\left[\frac{1-x^{m+1}}{m+1}\right]^{r}, \tag{3.8}$$

where c is a constant.

Therefore,

$$f_Y(x) = c\,x^{\gamma_{r+1}-1} + \frac{c_r\ x^{\gamma_{r+1}-1}}{\Gamma(r+1)}\left[\frac{1-x^{m+1}}{m+1}\right]^{r}. \tag{3.9}$$

We have from (3.9), $F_Y(0)=0$ and $F_Y(1) = \frac{c}{\gamma_{r+1}} + 1$. Since $f_Y(x)$ is a *pdf* with boundary condition $F_Y(0)=0$ and $F_Y(1)=1$, we must have $c=0$.

Hence, we obtain

$$f_Y(x) = \frac{c_r\ x^{\gamma_{r+1}-1}}{\Gamma(r+1)}\left[\frac{1-x^{m+1}}{m+1}\right]^r . \tag{3.10}$$

Thus, $Y = X'(r+1,n,m,k)$.

To prove (b) implies (a), we have

$$F_{l,r+1,n,m,k}(x) = P\{X'(r,n,m,k)\ W_{r+1} \le x\}$$

$$= \int_0^1 F_{l,r,n,m,k}\left(\frac{x}{u}\right) u^{\gamma_{r+1}-1}\gamma_{r+1}\,du$$

$$= x^{\gamma_{r+1}} + \gamma_{r+1}\int_r^1 F_{l,r,n,m,k}\left(\frac{x}{u}\right) u^{\gamma_{r+1}-1}\,du \quad .$$

Substituting $\frac{s}{u} = t$ in the integral, we obtain

$$F_{l,r+1,n,m,k}(x) = x^{\gamma_{r+1}} + \gamma_{r+1}x^{\gamma_{r+1}}\int_n^1 F_{l,r,n,m,k}(t)\left(\frac{1}{t}\right)^{\gamma_{r+1}+1} dt \tag{3.11}$$

Differentiating both sides of $w.r.t.\ x$, we obtain

$$f_{l,r+1,n,m,k}(x) = \gamma_{r+1}x^{\gamma_{r+1}-1} - \gamma_{r+1}x^{\gamma_{r+1}}F_{l,r,n,m,k}(x)x^{-\gamma_{r+1-1}}$$

$$+ (\gamma_{r+1})^2 x^{\gamma_{r+1}-1}\int_0^1 F_{l,r,n,m,k(t)}(t)\left(\frac{1}{t}\right)^{\gamma_{r+1}} dt$$

Thus, using (3.11), we obtain

$$f_{l,r+1,n,m,k}(x) = \gamma_{r+1}x^{\gamma_{r+1}-1} - \frac{F_{l,r,n,m,k}(x)}{x}$$

$$+ (F_{l,r+1,n,m,k}(x) - x^{\gamma_{r+1}})\gamma_{r+1}x^{-1} \tag{3.12}$$

On simplification, we obtain from (3.12)

$$f_{l,r+1,n,m,k}(x) = (-F_{l,r,n,m,k}(x) + F_{l,r+1,n,m,k}(x)\,\gamma_{r+1}x^{-1})$$

$$= \left(\frac{F(x)}{\gamma_{r+1} f(x)} f_{l,r+1,n,m,k}(x)\,\gamma_{r+1}x^{-1} \right) \tag{3.13}$$

Thus,

$$\frac{f(x)}{F(x)} = \frac{1}{x} \tag{3.14}$$

The solution of (3.14) with boundary condition $F(0) = 0$ and $F(1) = 1$ is

$$F(x) = x \ , \quad 0 \le x \le 1 . \tag{3.15}$$

The proof of Theorem for $m = -1$ is similar.

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